AMBIGUITIES IN MASS EXTRAPOLATION VIA PARTIALLY CONSERVED AXIAL-VECTOR CURRENTS AND THE $K\to 2\pi$ DECAYS*

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Following the success of the hypothesis of the partially convserved axial-vector current (PCAC)[1] and current commutation relations[2] for the calculation of the renormalized axial-vector coupling constant,[3] a number of applications have been made to determine relations between various weak decay rates.[4] These calculations all involve the evaluation of the matrix elements of an assumed weak Hamiltonian by the standard reduction techniques to bring one or more pion fields into the matrix element and extrapolate off the mass shell, and the use of PCAC and current commutation rules to relate various processess. (We shall refer to this entire formalism as PCAC.)

It is the purpose of this note to point out that if more than one pion is taken off the mass shell, there exist within this same framework alternative prescriptions which give different results, and to illustrate this by considering the $\Delta I = \frac{1}{2}$ rule for the $K\to 2\pi$ decay. The reduction technique relates matrix elements for states with various numbers of mass-shell particles, and replacing a pion field operator by the divergence of an axial current is, in itself, an identity on the mass shell. However, as will be illustrated below, if all relevant particles are reduced before the extrapolation, terms occur which are not present if reduction and extrapolation are carried out sequentially for one particle at a time. These extra terms, which can have different structure and even different selection rules, are essentially initial- or final-state interaction terms, and this suggests that the simultaneous reduction procedure is preferable. However, it should be observed that this is not known a priori and that an additional prescription within the method of PCAC is needed to select between the different extrapolations.

For the nonleptonic two-pion decays of the K mesons, it is shown that the previous conclusion,[5] that the $\Delta I = \frac{1}{2}$ rule is a consequence of a current-current form for the weak Hamiltonian and PCAC, depends upon the prescription, and that with an alternative and probably preferable procedure this is not a valid conclusion. Moreover, the experimental data for the $\Delta I \neq \frac{1}{2}$ parts of these decays are consistent with this latter hypothesis.

Let us consider the matrix element $K_{2\pi}$ for the $K\to 2\pi$ decay modes with an assumed current-current Hamiltonian H_W:

$$K_{2\pi} = \langle \pi^\alpha \pi^\beta | H_W(0) | K\rangle. \tag{1}$$

To simplify the equations, symmetrization will be implied but not explicitly included. The first procedure for obtaining the matrix elements at zero momentum for the pions prescribes taking one pion off the mass shell via PCAC before the second off-the-mass-shell extrapolation is made. This is the method used in the study of K decay by many authors[5,6] and is now summarized. Reducing one of the pions and using the relationship $\varphi_\pi(x) = C_\pi \partial_\mu A_\mu(x)$ between the pion field operator and the axial-vector current, one obtains (neglecting surface terms)

$$K_{2\pi}^{(0)} = i(C_\pi/\sqrt{2})\int d^4x\, e^{-iq^\alpha\cdot x}(\Box_x - \mu^2)\{iq_\mu^{\ \alpha}\langle \pi^\beta | \{A_\mu^{\ \alpha}(x), H_W\}_+ |K\rangle - \delta(x_0)\langle \pi^\beta | [A_0^{\ \alpha}(x), H_W] |K\rangle\}, \tag{2}$$

or in the limit as $q\to 0$, only the equal-time commutator survives:

$$\lim_{q\to 0} K_{2\pi}^{(0)} = i(C_\pi \mu^2/\sqrt{2})\langle \pi^\beta | [F_5^{\ \alpha}(0), H_W] |K\rangle, \tag{3}$$

where $F_5^{\ \alpha}(t) = \int d^3x\, A_0^{\ \alpha}(x)$. Equation (3) has been used to obtain a number of interesting relations among leptonic and nonleptonic K decays,[6] non of which are altered by the present work. Repeating this procedure for the second pion, one obtains the result[5] that the matrix element defined by this prescription,

$$K_{2\pi}^{(0)} \underset{\substack{q^\alpha \to 0\\ q^\beta \to 0}}{\longrightarrow} [(C_\pi \mu^2)^2/\sqrt{2}]\langle 0|[F_5^\beta,[F_5^\alpha,H_W]]|K\rangle, \quad (4)$$

satisfies the $\Delta I=\frac{1}{2}$ rule in the limit $q_\alpha \to q_\beta \to 0$. This has led to the conclusion[5] that PCAC with a current-current Hamiltonian leads to the $\Delta I=\frac{1}{2}$ rule since the essential assumption is that the values of the matrix elements at zero pion momentum are smoothly connected to the matrix elements on the mass shell.

An alternative procedure for the extrapolation of the $K_{2\pi}$ matrix element to zero pion momentum prescribes that the extrapolation be made for both pions simultaneously, rather than sequentially as in the first procedure:

$$K_{2\pi}^{(1)} = -(C_\pi^2/\sqrt{2})\int d^4x d^4y \exp(-iq_\alpha\cdot x - iq_\beta\cdot y)(\square_x-\mu^2)(\square_y-\mu^2)\langle 0|\{\partial_\mu A_\mu^\alpha(x), \partial_\nu A_\nu^\beta(y)H_W\}_+|K\rangle. \quad (5)$$

Under the same assumptions as above, one obtains

$$K^{(1)} = K_{2\pi}^{(0)} + [(C_\pi \mu^2)^2/\sqrt{2}]\int d^4x d^4y \exp(-iq_\alpha\cdot x)\exp(-iq_\beta\cdot y)\delta(x_0-y_0) \times\langle 0|\{[A_0^\alpha(x), \partial_\nu A_\nu^\beta(y)]H_W\}_+|K\rangle. \quad (6)$$

The additional term obtained in this second procedure for extrapolating off the mass shell contains the equal-time commutator called σ by Weinberg[7] in his Lagrangian formulation of the PCAC. In fact, the extrapolation defined in Eq. (5) is the one used by Weinberg in his treatment of pion scattering lengths, in which the σ contributions were discussed for pion-nucleon and pion-pion scattering.[8] In the evaluation of S-matrix elements, the ambiguity in the extrapolation procedure occurs when two or more pions are removed from the mass shell, as can be seen from Eqs. (2) and (5) with H_W replaced by another pion field. It should be noted that these extra σ terms cannot occur in the first procedure [Eq. (2)] since before reduction the pion is at infinite time and the σ terms arise from time ordering of the pion fields, so that symmetrization or surface terms will not alter these results.

Although the σ operator is assumed to be an isoscalar in Refs. 7 and 8, both $I=0$ and $I=2$ operators occur, an occurence which is essential to the discussion of K decay which follows. Arguments are given in Ref. 7 for dropping the σ terms in pion-nucleon scattering on the basis of the σ model and from the structures of the σ operator (i.e., the operator $\partial_\mu A_\mu$ presumably introduces a factor of μ^2), while the $I=0$ σ term is included in the derivation of the pion-pion scattering lengths. We would like to emphasize that the importance of these terms in any process is intimately connected with the magnitude at the pion-pion interaction in the $I=0$ and $I=2$ states, which is not very well known now. The two-pion K decay is discussed next as an interesting example of the possible relevance of these ambiguous terms. Of course, additional terms also appear in $K\to 3\pi$ decays when two or more pions are taken off the mass shell, and previous results must also be suitably modified. These latter are not discussed here.

The prediction of the first procedure, Eq. (4), for the $K\to 2\pi$ decays is that the decay $K^+ \to \pi^+ + \pi^0$ is forbidden, while the decays $K_S \to \pi^+ + \pi^-$ or $\to \pi^0 + \pi^0$ satisfy the $\Delta I=\frac{1}{2}$ rule. However, from Eq. (6), it is obvious that the $I=2$ σ operator will lead to a $\Delta I=\frac{3}{2}$ decay, and so the decay rate for $K^+\to\pi^+ + \pi^0$ can be related to the deviation of the branching ratio of $K_S\to \pi^+ + \pi^-$, compared to $K_S \to \pi^0 + \pi^0$.

A simple phenomenological analysis shows that the $\Delta I \neq \frac{1}{2}$ parts are consistent. For this it is necessary to consider only $I=0$ and $I=2$ pion states, so that there are two matrix elements, say, M_0 and M_2 for decay into the $I=0$ and $I=2$ pion states, respectively. The magnitude of M_2 can be determined from the decay rate of $K^+\to\pi^+ + \pi^0$, while the M_0 matrix element can be approximately determined from the total $K\to 2\pi$ rate. The experimental branching ratio of $(K_S\to \pi^+ + \pi^-)/(K_S\to\pi^0+\pi^0)$ is larger than the $\Delta I=\frac{1}{2}$ prediction. The experimental value for $\mathrm{Re}[M_0{}^*M_2]$ can be determined from this deviation. It turns out that $\mathrm{Re}M_0{}^*M_2$ from the K_S experimental branching ratio is about the same size as $|M_0||M_2|$, so the $K\to 2\pi$ decays are consistent with a small or moderate phase difference in the two states. The numerical result is that $\cos(\varphi_2-\varphi_0)=\mathrm{Re}[M_0{}^*M_2]/|M_0||M_2| \cong 1.0\pm 0.2$, where $\varphi_2-\varphi_0$ is the phase difference for the two matrix elements.[9] Thus, these decays are consistent with the $\Delta I\neq\frac{1}{2}$ terms arising from the $I=2$ σ operator. Of course, there are great uncertainties arising from the large momentum of the pions in the $K\to 2\pi$ decays.

Although it does not follow from first principles that this second procedure is the correct one, the fact that the extra terms that appear correspond to π-π interactions suggests that they have a sound physical basis and that this is the proper procedure. Since the matrix elements of the σ operators which occur in the K decays discussed here also appear in a study of π-π scattering, it should be fruitful to test the consistency between these processes. A study of the $I=2$ π-π channel should be particularly interesting.[10]

In conclusion, we see that the $\Delta I=\frac{1}{2}$ rule for K decay is not a consequence of the current-current Hamiltonian and PCAC, for it only turns out a posteriori that the matrix elements of operators violating $\Delta I=\frac{1}{2}$ are small. Thus, there seem to be deeper dynamical reasons for the $\Delta I=\frac{1}{2}$ rule, which, in turn, result in the ambiguities in PCAC being numerically unimportant in the nonleptonic decays. Whether this takes care of these amibguities in every process is yet to be seen, and suggests the value of the study of σ terms in individual processes.

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O(4) SYMMETRY AND REGGE-POLE THEORY*

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Recent theoretical work has indicated that Regge trajectories really occur in families with definite requirements on the spacing of trajectories and the behavior of residue functions at zero values of the invariant mass. Such results have been derived from analyticity arguments for nucleon-nucleon scattering[1] and for unequal-mass processes.[2,3]

The purpose of this note is to study a more general and elegant explanation of these families of trajectories which has been suggested by several authors.[2,4,5] According to this explanation, the existence of Regge families follows because, as a general consequence of Lorentz invariance, scattering amplitudes at zero values of the Mandelstam invariants have a special invariance property. Consider the process in which momenta $p_1+p_2\to p_1'+p_2'$. Let $K=p_1+p_2$ denote the total energy-momentum vector and take $t=K^2$. The invariance group