ELECTROMAGNETIC MASS DIFFERENCE OF PIONS*†

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Using the algebra of currents and the sum rules recently derived by Weinberg, we calculate the electromagnetic mass difference of pions and obtaim $m_{\pi^+}-m_{\pi^0}\simeq 5.0$ MeV.

Recently Weinberg[1] has obtained sum rules for the spectral functions of vector and axial-vector current propagators, from which he has derived a relation between the masses of ρ and A_1, in remarkable agreement with experiment. The purpose of this note is to show that using similar techniques, one can calculate the electromagnetic mass splitting of the pions. We use the soft-pion technique to reduce the virtual photon-pion Compton scattering amplitude in terms of the propagator functions of the vector and axial-vector currents. Since, as we shall show in Eq. (12), the difference $m_{\pi^+}{}^2-m_{\pi^0}{}^2$ is independent of the mass of the pion, we have an a posteriori justification for the use of the soft-pion technique. Although we have used partially conserved axial-vector currents (PCAC)[2] in the reduction technique and subsequently employ Weinberg's sum rules which hold when the axial-vector currents are divergenceless, this apparent inconsistency may be resolved if we take either of the following two viewpoints: (i) After the soft-pion reduction, since the expression for $m_{\pi^+}{}^2-m_{\pi^0}{}^2$ is independent of the pion mass, we may at this stage set $m_\pi=0$; or (ii) for a realistic pion mass, Weinberg's sum-rules would presumably[1] be altered only by terms of the order of $(m_\pi/m_\rho)^2$, which may be neglected. If we assume that the propagator functions are dominated by the low-lying poles, namely ρ, π, and A_1, our calculation yields a value of $m_{\pi^+}-m_{\pi^0}\simeq 5.0$ MeV.

The electromagnetic self-energy of the pion in the second order is given by[3]

$$\Delta E=(2\pi)^3e^2\,\mathrm{Re}(1/2i)\int d^4x\,\langle 0|T[a_\mu(x)a_\nu(0)]|0\rangle\,\{\langle\pi|T[V_\mu^{\mathrm{em}}(x)V_\nu^{\mathrm{em}}(0)]|\pi\rangle-\langle 0|T[V_\mu^{\mathrm{em}}(x)V_\nu^{\mathrm{em}}(0)]|0\rangle\},\quad(1)$$

where we have dropped the contact terms proportional to $\varphi^2 a_\mu a_\mu$, since the contribution due to these terms can be made to vanish by the choice of a suitable gauge.[4] In this gauge, the difference of the squared masses of π^+ and π^0 is given by

$$m_{\pi^+}{}^2-m_{\pi^0}{}^2=-\frac{e^2}{4\pi}2m_\pi\,\mathrm{Re}\left[\int\frac{d^4q}{q^2-i\epsilon}\left(\delta_{\mu\nu}-4\frac{q_\mu q_\nu}{q^2}\right)\right]$$
$$\times\int d^4x\,e^{iq\cdot x}\{\langle\pi^+|T[V_\mu^{\mathrm{em}}(x)V_\nu^{\mathrm{em}}(0)]|\pi^+\rangle-\langle\pi^0|T[V_\mu^{\mathrm{em}}(x)V_\nu^{\mathrm{em}}(0)]|\pi^0\rangle\}.\quad(2)$$

Using the soft-pion technique with PCAC[2] and the equal-time commutation relations,[5] we obtain the expression given by[6]

$$m_{\pi^+}{}^2-m_{\pi^0}{}^2=\frac{e^2}{(2\pi)^4}\frac{4m_\pi{}^4}{c_\pi{}^2}\int\frac{d^4q}{q^2-i\epsilon}\left(\delta_{\mu\nu}-4\frac{q_\mu q_\nu}{q^2}\right)[\Delta_{\mu\nu}{}^V(q)-\Delta_{\mu\nu}{}^A(q)],\quad(3)$$

where

$$\Delta_{\mu\nu}^{V}(q)=\int d^4x\, e^{iq\cdot x}\langle 0|T[V_\mu^{3}(x)V_\nu^{3}(0)]|0\rangle \tag{4}$$

and a similar expression for $\Delta_{\mu\nu}^{A}(q)$ with axial-vector currents.

Using the expressions for the two-point functions of the vector and the axial-vector currents given by Weinberg,[1] we obtain the following spectral representations of the propagator functions[7]:

$$\langle 0|T[V_\mu^{3}(x)V_\nu^{3}(0)]|0\rangle=\frac{-i}{4(2\pi)^4}\int_0^\infty dm^2\rho_\nu(m^2)\int d^4p\, \times e^{ip\cdot x}\left(\delta_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}\right)\frac{1}{p^2+m^2-i\eta}+\frac{i}{4}\delta_{\mu4}\delta_{\nu4}\delta^{(4)}(x)\int_0^\infty dm^2\frac{\rho_V(m^2)}{m^2}, \tag{5}$$

and

$$\langle 0|T[A_\mu^{3}(x)A_\nu^{3}(0)]|0\rangle=\frac{-i}{4(2\pi)^4}\int_0^\infty dm^2\rho_A(m^2)\int d^4p\, e^{ip\cdot x}\left(\delta_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}\right)\frac{1}{p^2+m^2-i\eta}+\frac{i}{4}\delta_{\mu4}\delta_{\nu4}\delta^{(4)}(x)\int_0^\infty dm^2\frac{\rho_A(m^2)}{m^2}-\frac{-i}{4(2\pi)^4}F_\pi^{\,2}\int d^4p\, e^{ip\cdot x}\frac{p_\mu p_\nu}{p^2-i\eta}+\frac{i}{4}\delta_{\mu4}\delta_{\nu4}\delta^{(4)}(x)F_\pi^{\,2}, \tag{6}$$

where the spectral functions $\rho_V(m^2)$ and $\rho_A(m^2)$ satisfy the following sum rules[1]:

$$\int_0^\infty\{\rho_V(m^2)-\rho_A(m^2)\}dm^2=0 \tag{7}$$

and

$$\int_0^\infty\frac{\rho_V(m^2)-\rho_A(m^2)}{m^2}dm^2=F_\pi^{\,2}, \tag{8}$$

with

$$F_\pi=C_\pi/m_\pi^{\,2}.$$

Substituting Eqs. (5), (6), and (8) in Eq. (3), we obtain

$$m_{\pi^+}^{\,2}-m_{\pi^0}^{\,2}=-\frac{3ie^2}{(2\pi)^4}\frac{m_\pi^{\,4}}{C_\pi^{\,2}}\int\frac{d^4q}{q^2}\int_0^\infty dm^2\frac{\rho_V(m^2)-\rho_A(m^2)}{q^2+m^2}. \tag{9}$$

We now approximate[1] the spectral functions by retaining only the ρ and A_1 poles, i.e.,

$$\rho_V(m^2)=g_\rho^{\,2}\delta(m^2-m_\rho^{\,2}),$$
$$\rho_A(m^2)=g_A^{\,2}\delta(m^2-m_A^{\,2}). \tag{10}$$

Equation (9) with Eqs. (7) and (8) now leads to[8]

$$m_{\pi^+}^{\,2}-m_{\pi^0}^{\,2}=(3\ln 2)\frac{e^2}{4\pi}\frac{m_\pi^{\,4}}{C_\pi^{\,2}}\frac{g_\rho^{\,2}}{4\pi}. \tag{11}$$

Using the value of $g_\rho^{\,2}$ from either current algebra[9] or the observed ρ width, we have $g_\rho^{\,2}\simeq 2m_\rho^{\,2}F_\pi^{\,2}$, so that we finally get, from Eq. (11),

$$m_{\pi^+}^{\,2}-m_{\pi^0}^{\,2}=\frac{(3\ln 2)}{2\pi}\frac{e^2}{4\pi}m_\rho^{\,2}, \tag{12}$$

which gives the mass difference

$$m_{\pi^+}-m_{\pi^0}\simeq 5.0\text{ MeV},$$

in reasonable agreement with the experimental value[10] of 4.6 MeV. It is not surprising that the approximation of keeping only the lowest lying resonances in the spectral functions leads to a reasonable value of the π^+-π^0 mass difference, since the $\Delta I = 2$ part of the effective electromagnetic interaction relevant in this case is well described[11] by low q^2 values.

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[8]Note that because of the sum rule (7) we have $g_\rho{}^2 = g_A{}^2$, which plays a crucial role in providing the convergence of the integral over g^2 in (9).

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SYMMETRY, SUPERCONVERGENCE, AND SUM RULES FOR SPECTRAL FUNCTIONS*

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We discuss the convergence and the superconvergence properties of the invariant amplitudes which occur in suitable combinations of the propagator functions of the vector and the axial-vector currents, based on the use of symmetry arguments for the asymptotic behavior. The sum rules so obtained for the spectral functions are in good agreement with experiment.

In this note we discuss a general way to obtain sum rules for the spectral functions of vector and axial-vector currents. Two of the sum rules we derive have been recently obtained by Weinberg[1] under the more restrictive assumption that the pion is massless, so that the axial-vector current is divergenceless. Our method exploits the possibility of superconvergence of amplitudes occurring in suitable combinations of the vector and axial-vector two-point functions. We show that the case for convergence or superconvergence can be made on the basis of group-symmetry arguments alone. This provides some interesting insight into the relation between the concepts of superconvergence and group symmetry. Finally, we show that the sum rules we so obtain are well saturated by suitable low-lying particle states.

We start by considering the vacuum expectation values of the time-ordered product of two vector and two axial-vector currents, whose Fourier transforms are given, respectively, by

$$\Delta_{\mu\nu}{}^V(q) = \int d^4x\, e^{-iq\cdot x}\langle 0|T\{V_{\mu j}{}^i(x)V_{\nu i}{}^j(0)\}|0\rangle, \quad (1)$$

$$\Delta_{\mu\nu}{}^A(q) = \int d^4x\, e^{-iq\cdot x}\langle 0|T\{A_{\mu j}{}^i(x)A_{\nu i}{}^j(0)\}|0\rangle, \quad (2)$$

where i and j are the SU(2) indices ($i, j = 1, 2$). If SU(2)⊗SU(2) were an exact symmetry, we would have $\Delta_{\mu\nu}{}^V(q) = \Delta_{\mu\nu}{}^A(q)$ for all values of q. However, in nature this symmetry seems to be broken, but nevertheless one expects that in the asymptotic limit $q \to \infty$, SU(2)⊗SU(2) would