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THEORY OF FIRST SOUND IN DILUTE SOLUTIONS OF He³ IN He⁴ AT VERY LOW TEMPERATURES*

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The velocity and attenuation of first sound in dilute solutions of He³ in very low temperature He⁴ are calculated. The theoretical attenuation agrees well with that measured recently by Abraham, Eckstein, Ketterson, and Vignos.

Introduction of He³ atoms into superfluid He⁴ modifies the first-sound velocity and, by providing additional degrees of freedom for energy absorption, increases the attenuation of first sound. At very low temperatures the attenuation is due entirely to the viscosity of the He³ and can, as this paper will show, be calculated precisely. This is because, on the one hand, the exact matrix element for the coupling of a long wavelength phonon to a He³ quasiparticle can be deduced from a combination of thermodynamic and Galilean invariance arguments, and on the other hand, the He³-He³ scattering time τ_η for viscosity can be calculated from the effective He³-He³ interaction $v(\vec{k})$, determined by Bardeen, Baym, and Pines.¹ Measurement of the attenuation of first sound at very low temperatures thus provides a direct measure of τ_η ; the attenuation calculated here agrees well with the recent measurements of Abraham, Eckstein, Ketterson, and Vignos.²

At very low temperatures attenuation due to phonon-phonon scattering rapidly approaches zero. Also, since the Fermi velocity of the He³ is always much smaller than the first-sound velocity, damping of sound by single-particle excitations from the Fermi sea is forbidden by energy conservation. Furthermore, atten-

uation of sound via "Compton scattering" of phonons with He³ atoms is exceedingly slow. The collision time τ_c for this process is given roughly by³ $\omega\tau_c \sim 10^{36}/\omega^3 x$, where x is the concentration of He³. In the ultrasonic region $\omega\tau_c$ is many orders of magnitude greater than one. The primary mechanism, then, for absorption of sound is through the He³ longitudinal viscosity η , that is, through the collisions of the He³, which are carried along in a first-sound wave, with themselves. The collision time τ_η for this process, calculated^{4,5} from the effective interaction $v(\vec{k})$ between He³ atoms,¹ is

$$\tau_\eta = (17.8 \times 10^{-12}/T^2) \text{ sec } (^{\circ}\text{K})^2,$$

in a solution of concentration $x = 0.05$ at low temperature.

The shift in velocity and the attenuation of first sound in mixtures can be calculated by using conservation laws combined with the He³ kinetic equation. At very low temperatures ($\leq 0.2^{\circ}\text{K}$) thermal excitations of the He⁴ may, to a first approximation, be neglected. The collective excitations of the mixture are then described by the variables $\rho_4(\vec{r}t)$, the local He⁴ particle density, $\vec{v}_s(\vec{r}t)$, the local superfluid velocity, and $n_p^*(\vec{r}t)$, the He³ quasiparticle dis-

tribution function. The He^3 quasiparticles behave as normal fluid; if we let m be the He^3 effective mass evaluated at zero concentration, then each He^3 atom carries with it a mass $\delta m = m - m_3$ of He^4 in its screening cloud. [m_3 is the He^3 atomic mass; from Ref. 1, $m = 2.34m_3$.] Thus the superfluid mass density is $m_4\rho_4(\vec{r}t) - \delta m\rho_3(\vec{r}t)$, where ρ_3 is the local He^3 particle, or equivalently, quasiparticle density. Since He^3 quasiparticles are conserved, the superfluid mass density obeys the (linearized) conservation law

$$(\partial/\partial t)(m_4\rho_4 - \delta m\rho_3) + m_4 n_s \nabla \cdot \vec{v}_s = 0; \quad (1)$$

$m_4 n_s = m_4 n_4 - \delta m n_3$ is the equilibrium superfluid mass density, and n_4 and n_3 are the equilibrium He^4 and He^3 particle densities. Furthermore, the superfluid velocity obeys the equation of motion

$$m_4 \partial \vec{v}_s / \partial t = -\nabla \mu_4(\rho_4, \rho_3) \\ = -(\partial \mu_4 / \partial n_4)_{n_3} \nabla \rho_4 - (\partial \mu_4 / \partial n_3)_{n_4} \nabla \rho_3, \quad (2)$$

where μ_4 is the local He^4 chemical potential; external driving forces on the He^4 are added onto the right side of this equation.

To describe the propagation of first sound, we must then determine how the He^3 density is driven by superfluid velocity and He^4 density fluctuations. The energy of a He^3 quasiparticle, when $\vec{v}_s = 0$, may be written in the form

$$\epsilon_{\vec{p}} = \epsilon_0(\rho_3, \rho_4) + p^2/2m - \sum_{\vec{p}'} v(\vec{p} - \vec{p}') n_{\vec{p}'}. \quad (3)$$

ϵ_0 is the effective local field felt by the quasiparticles; to order x^0 in equilibrium^{6,7} ($\partial \epsilon_0 / \partial n_4)_{n_3} = (m_4 s^2 / n_4)(1 + \alpha) = (\partial \mu_4 / \partial n_3)_{n_4}$ and $(\partial \epsilon_0 / \partial n_3)_{n_4} = (m_4 s^2 / n_4)(1 + 2\alpha)$, where s is the first-sound velocity (~ 238 m/sec) of pure He^4 and $\alpha = 0.28$ is the fractional excess molar volume of dilute He^3 in He^4 . The last term in (3) is the exchange energy; $v(\vec{p} - \vec{p}')$ is the effective interaction between He^3 quasiparticles,¹ and the sum over \vec{p}' is restricted to particles whose spin is parallel to the particle whose energy we are considering. All Fermi liquid effects are accounted for in ϵ_0 and this exchange term.

The dependence of $\epsilon_{\vec{p}}$ on the local He^4 density is through ϵ_0 ; the dependence of m on the local He^4 density leads to corrections $\sim v_F^2/s^2 \sim 0.01$, where v_F is the He^3 Fermi velocity. To determine the dependence of $\epsilon_{\vec{p}}$ on \vec{v}_s we assume that $\vec{v}_s \neq 0$; then a quasiparticle of mo-

mentum \vec{p} in the frame where $\vec{v}_s = 0$ has energy given by (3); in the lab frame its energy is given, to first order in \vec{v}_s , by $\epsilon_{\vec{p}} + \vec{p} \cdot \vec{v}_s$, while its momentum is $\vec{p} + m_3 \vec{v}_s$. Thus, in the presence of a superfluid velocity,⁸

$$\epsilon_{\vec{p}}(\vec{v}_s) = \epsilon_{\vec{p}} + (\delta m/m) \vec{p} \cdot \vec{v}_s. \quad (4)$$

[Fermi liquid corrections do not appear in the m in this equation.]

The linear response, $\delta \rho_3 = \rho_3 - n_3$, of the He^3 to He^4 motion of frequency ω and wave number k can then be written as

$$\delta \rho_3 = \langle \rho_3 \rho_3 \rangle [(\partial \mu_4 / \partial n_3)_{n_4} \delta \rho_4 + \omega \delta m v_s / k], \quad (5)$$

where $\delta \rho_4 = \rho_4 - n_4$ and $\langle \rho_3 \rho_3 \rangle(k, \omega)$ is the Fourier transform of the retarded He^3 density-density correlation function. To calculate this function we solve the He^3 Boltzmann equation, including He^3 - He^3 collisions by writing a relaxation-time approximation that conserves quasiparticle number and momentum. In the limit $\omega \gg kv_F$ the result is

$$\langle \rho_3 \rho_3 \rangle = \frac{k^2 n_3}{m \omega^2 + ik^2 \omega \eta / 3n_3 (1 + \tau \omega^2)}. \quad (6)$$

He^3 compressibility terms in the denominator, of relative order $(kv_F/\omega)^2$, have been omitted, as have Fermi liquid effects beyond F_1 . The He^3 longitudinal viscosity η is given in terms of the He^3 relaxation time τ_η by $\eta = \frac{4}{3} m^* v_F^2 \times n_3 \tau_\eta Y(T)$, where m^* is the He^3 effective mass at concentration x . The monotonically increasing function $Y(T)$ is given at low T by

$$Y(T) = 1 + (32\pi^2/35)(T/T_F)^2 - (15\pi^4/128)(T/T_F)^4,$$

and generally by

$$Y(T) = 1 + \frac{3}{2}(P_T/P_0 - 1) + (3/4P_0) \int_0^T S_T dT',$$

where P_T and S_T are the pressure and entropy density of a free Fermi gas of effective mass m^* and density n_3 at temperature T .

Fourier transforming Eqs. (1) and (2), using (1) to eliminate v_s and substituting Eq. (5) into (2), we find that the dispersion relation for first sound in the mixture is given by the solution of the equation

$$m_4 \omega^2 = n_s k^2 \{ (\partial \mu_4 / \partial n_4)_{n_3} \\ + \langle \rho_3 \rho_3 \rangle [(\partial \mu_4 / \partial n_3)_{n_4} + \delta m \omega^2 / n_s k^2] \}. \quad (7)$$

In writing (7) we have neglected a correction to the final term of relative order x . The term $(\partial \mu_4 / \partial n_4)_{n_3}$ depends on x . The quantity in square brackets is the effective matrix element for the coupling of a He^3 quasiparticle to a long wavelength He^4 phonon. This matrix element is determined exactly by thermodynamic and Galilean invariance arguments; in the limit $x \rightarrow 0$ and for $\omega = sk$ it becomes

$$(m_4 s^2 / n_4)(1 + \alpha + \delta m / m_4) = 2.28 m_4 s^2 / n_4.$$

From (7) we find that the first-sound velocity s_x in the mixture is given by⁹

$$\begin{aligned} s_x / s = 1 + \frac{1}{2} x [& (m_4 / m)(1 + \alpha + \delta m / m_4)^2 \\ & - (1 + \alpha + \delta m / m_4) + n_4 \partial \alpha / \partial n_4] \\ \approx 1 - 0.23x. \end{aligned} \quad (8)$$

The first term in the square brackets is an increase in the sound velocity due to the interaction of the He^4 with the He^3 ; the next two terms, which are both negative, represent a decrease in the sound velocity due to the lowering of the He^4 density, and hence partial compressibility, caused by the addition of He^3 .

The amplitude attenuation α_I of first sound, determined by the imaginary part of the solution to (7), is given by

$$\alpha_I = \frac{x}{15} \frac{v}{s^3} \frac{F^2}{m^*} \frac{m_4}{m} (1 + \alpha + \delta m / m_4)^2 Y(T) \frac{2\omega^2 \tau \eta}{1 + \omega^2 \tau^2 \eta^2}. \quad (9)$$

The concentration and frequency dependence of this result is similar to that derived by Eckstein¹⁰ by similar methods. For a 5% solution,

$$\alpha_I = (0.57 \times 10^{-8}) Y(T) 2\omega^2 \tau \eta / (1 + \omega^2 \tau^2 \eta^2), \quad (10)$$

in cgs units. Since $\tau \eta$ has been calculated, all the parameters in the formula for α_I are known.

The low-temperature attenuation for a 5% solution at frequencies of 20, 60, 100, and 140 Mc/sec is shown in Fig. 1. The data points [squares at 20 Mc/sec, circles at 60 Mc/sec] are those of Abraham *et al.*² as given in the figure of their paper. At 20 Mc/sec both the theoretical and experimental maxima occur at 0.052°K, while the maximum theoretical attenuation is about 5% greater than the experimental value. The theory agrees closely with the experiment at 20 Mc/sec up to $T \sim 0.1^\circ\text{K}$. The larger experimental values of α_I above that temperature indicate that $\tau \eta T^2$ begins to

increase with temperature. [In fact, $\tau \eta T^2$ approaches the theoretical value $74 \times 10^{-12} \text{ sec } (^\circ\text{K})^2$ in the nondegenerate regime.⁴] Furthermore, attenuation via phonon-phonon scattering begins to play a role above about 0.15°K.

Below the 20-Mc/sec curve we have plotted, as a dash-dot line, formula (10) at 20 Mc/sec replacing $Y(T)$ by 1, its $T=0$ value; the maximum is decreased and shifted to $T=0.047^\circ\text{K}$. This is the expected form of the attenuation at lower frequencies and temperatures.

The 60-Mc/sec data fall about 0.6 cm^{-1} below the theoretical curve. The dashed line through these data is the 60-Mc/sec theoretical line reduced by 20%. The temperature dependence of the data agrees very well with the theory; as in the 20-Mc data, one sees that $\tau \eta T^2$ begins to increase above 0.1°K . The comparison between theory and experiment at 100 and 140 Mc/sec is similar to that at 60 Mc/sec. Note how at these higher frequencies α_I begins, as

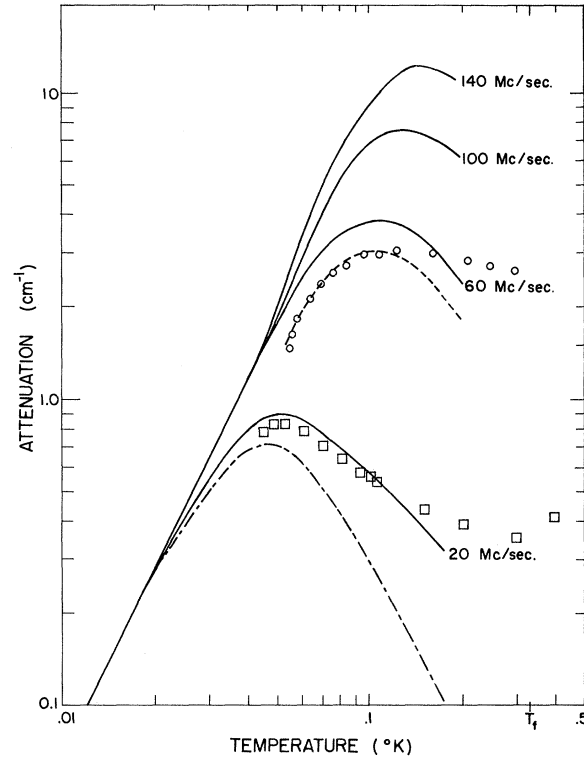


FIG. 1. Calculated attenuation of first sound in a 5% solution of He^3 in He^4 . The solid lines are the calculated attenuation at 20, 60, 100, and 140 Mc/sec. The dashed-dotted line at the bottom is the attenuation at 20 Mc/sec calculated with the factor $Y(T)$ replaced by 1. The data points are those of Abraham *et al.*²; the squares are their 20-Mc/sec data, while the circles are their 60-Mc/sec data. The line drawn through the 60-Mc/sec data is the theoretical attenuation reduced by 20%.

a consequence of the factor $Y(T)$, to increase faster than T^2 above 0.05°K; this feature is observed experimentally. When the additional attenuation due to phonon-phonon scattering is added to that due to He³ viscosity, the fall-off of the attenuation beyond 0.15°K becomes less pronounced than in Fig. 1.

The discrepancies between the present theory and the experiments are not crucial for two reasons. First, in this calculation, terms of relative order x have been neglected; thus one expects Eq. (9) for a 5% solution to be accurate at most to 5 or 10%. Second, the experiments measured only attenuation differences at each frequency. The absolute normalization of the data, i.e., the zero of attenuation at each frequency, was inferred indirectly and the uncertainties in this procedure are a possible source of discrepancy.

It is a pleasure to thank Professor David Pines, Professor John Wheatley, and Mr. Charles Ebner for several useful discussions concerning this calculation, and to express my gratitude to Dr. Bernard Abraham for an enlightening discussion about the experiments.

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¹J. Bardeen, G. Baym, and D. Pines, Phys. Rev. Letters 17, 372 (1966).

²B. M. Abraham, Y. Eckstein, J. Ketterson, and J. Vignos, Phys. Rev. Letters 17, 1254 (1966).

³This estimate is that given by H. London, G. R. Clarke, and E. Mendoza [Phys. Rev. 128, 1992 (1962)] from the data of D. J. Sandiford and H. A. Fairbank, in Proceedings of the Seventh International Conference on Low-Temperature Physics, Toronto, 1960, edited by G. M. Graham and A. C. Hollis Hallett (University of Toronto Press, Toronto, Canada, 1961), p. 641, evaluated for a thermal phonon.

⁴C. Ebner, private communication.

⁵R. Roach, dissertation, University of Illinois, 1966 (unpublished).

⁶J. Bardeen, G. Baym, and D. Pines, to be published.

⁷G. Baym, Phys. Rev. Letters 17, 952 (1966); ϵ_0 is given by $(\partial E/\partial n_3)_{n_4}$, where E is the ground-state energy per unit volume of the mixture minus the fermion exchange energy.

⁸Note how this result differs from the usual energy shift $\tilde{p}\cdot\tilde{v}_S$ for a phonon [I. M. Khalatnikov, Introduction to the Theory of Superfluidity (W. A. Benjamin, Inc., New York, 1965)]. Indeed, if the He³ did not interact with the He⁴, then $\delta m = 0$ and the coupling (4) would vanish.

⁹The shift in the first-sound velocity is in accord with that calculated by Khalatnikov, Ref. 8, Eq. (24-73).

¹⁰S. Eckstein, Phys. Rev. Letters 17, 1257 (1966). Among the shortcomings of this paper are that it fails to account correctly for the coupling (4) of the He³ to the superfluid velocity arising from Galilean invariance, or for effects of the He³ on the superfluid acceleration. As a consequence the magnitude of the attenuation that one would derive from that paper differs substantially from (9).

ULTRASONIC ATTENUATION IN A PURE TYPE-II SUPERCONDUCTOR NEAR H_{C2}

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One of us^{1,2} has recently proposed a new theory of ultrasonic attenuation in the mixed state of a pure type-II superconductor in a high magnetic field. The purpose of this Letter is to present recent experimental results on longitudinal and transverse wave propagation in two pure single crystals of niobium and to compare these results with the theoretical predictions. The crystals were oriented in the [111] direc-

tion and in the [100] direction and had resistivity ratios of $\rho_{300}/\rho_{4.2} \approx 150$ and ≈ 300 , respectively.

The theory is developed for circumstances where the mean free path of the electron is much larger than the coherence length, $l/\xi_0 \gg 1$, and where the upper critical field $H_{C2} \gg H_0 - H_0$ (H_0 is the applied external magnetic field). In the case where the wave vector ql is paral-