SHAPE OF HEAVY NUCLEI*

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(Received 10 March 1967)

This Letter reports some estimates of the binding energies of systems of very many nucleons in several geometric configurations by means of the Bethe-Weiszäcker semiempirical mass formula. The configurations investigated were a spherical shell, and oblate and prolate spheroids. It was found that, for beta-stable nuclei with more than 104 protons, the most energetically favorable configuration is a prolate spheroid.

The semiempirical mass formula[1] gives the total energy of a spherical nucleus of A nucleons, with Z protons and $N=A-Z$ neutrons, as

$$E=-f_1(A,Z)A+f_2(A,Z)A^{2/3}+C(A,Z). \qquad (1)$$

Here $C(A,Z)$ is the Coulomb energy, well represented by $\frac{3}{5}(Ze)^2/r_C A^{1/3}$. f_1 is supposed to represent the binding energy per nucleon in infinite nuclear matter; f_2 is the energy necessary to form a surface and expresses chiefly the reduced binding of nucleons near the surface. f_1 and f_2 are taken to depend on the square of the neutron excess $D=N-Z$; they are commonly expanded in powers of $(D/A)^2$. Considerable progress has been made in calculating f_1 and f_2 from the theory of nuclear matter. However, the best knowledge of these functions is from fits to experimental data on the binding energy of nuclei. Green,[2] writing $f_1=a_1-\alpha_4(D/A)^2$ and $f_2=a_2-\alpha_5(D/A)^2$, finds $a_1=15.88$, $a_2=17.97$, $\alpha_4=31.5$, $\alpha_5=40.0$, and $r_C=1.216$ (all energies in MeV, distances in 10^{-13} cm). Nemeth[3] has been able to calculate the symmetry energy from nuclear matter theory; she finds values for f_1 which agree well with the empirical coefficients given by Green and shows that the semiempirical formula should continue to be valid up to quite large asymmetries.

The expression (1) for the nuclear binding energy is adapted to nonspherical geometry by supposing that the term $f_2A^{2/3}$ is simply proportional to the area of the nuclear surface. Thus the surface energy becomes $f_2A^{2/3}g_5$, where g_5 is the ratio of the surface area of the configuration investigated to that of a sphere of equal volume. The Coulomb energy may, of course, be calculated from Poisson's equation; the result can again be written as $g4\frac{3}{5}[(Ze)^2/r_C]\times A^{-1/3}$, where g_4 is the ratio of the Coulomb energy to that of a sphere of equal volume.

The beta-stability condition $\partial E/\partial N=\partial E/\partial Z$ gives for the beta-stable value of Z

$$Z=\frac{\alpha_4A^{1/3}-\alpha_5g_5}{[(2/A)(\alpha_4A^{1/3}-\alpha_5g_5)+\frac{3}{10}(e^2/r_C)g_4]}. \qquad (2)$$

For the geometries investigated here, g_4 and g_5 are functions of a single deformation parameter x, so that the most stable configuration is characterized by $\partial E/\partial x=0$.

The shapes investigated here include a prolate spheroid, an oblate spheroid, and a spherical shell of uniform density from radius $R-\frac{1}{2}t$ to $R+\frac{1}{2}t$, and zero density elsewhere. The deformation parameter x for the spheroids is taken as the distance between the foci of the rotated ellipse divided by its major axis. For the spherical shell, x is the ratio $\frac{1}{2}t/R$ of the thickness to twice the arithmetic mean radius. Assuming a uniform density of charge, it is straightforward to calculate g_4 and g_5. They are, for a prolate spheroid,

$$g_4=\frac{1}{2x}(1-x^2)^{1/3}\ln\frac{1+x}{1-x}, \qquad (3a)$$

$$g_5=\tfrac{1}{2}(1-x^2)^{1/3}\left(1+(1-x^2)^{-1/2}\,\frac{\arcsin x}{x}\right). \qquad (3b)$$

For an oblate spheroid,

$$g_4=\frac{1}{x}(1-x^2)^{1/6}\arcsin x, \qquad (4a)$$

$$g_5=\tfrac{1}{4}(1-x^2)^{2/3}\left(\frac{2}{1-x^2}+\frac{1}{x}\ln\frac{1+x}{1-x}\right). \qquad (4b)$$

For a spherical shell,

$$g_4=5\left(\frac{x}{36}\right)^{1/3}\frac{(1-\frac{1}{3}x+\frac{1}{3}x^2+\frac{1}{15}x^3)}{(1+\frac{1}{3}x^2)^{5/3}}, \qquad (5a)$$

$$g_5=\frac{2}{x}\left(\frac{x}{36}\right)^{1/3}\frac{(1+x^2)}{(1+\frac{1}{3}x^2)^{2/3}}. \qquad (5b)$$

The total energies of the stable configurations for a given nucleon number A are plotted in Fig. 1 as A ranges from 300 to 1500. These are both beta stable and deformation stable, except that the total energy of a sphere, giv-

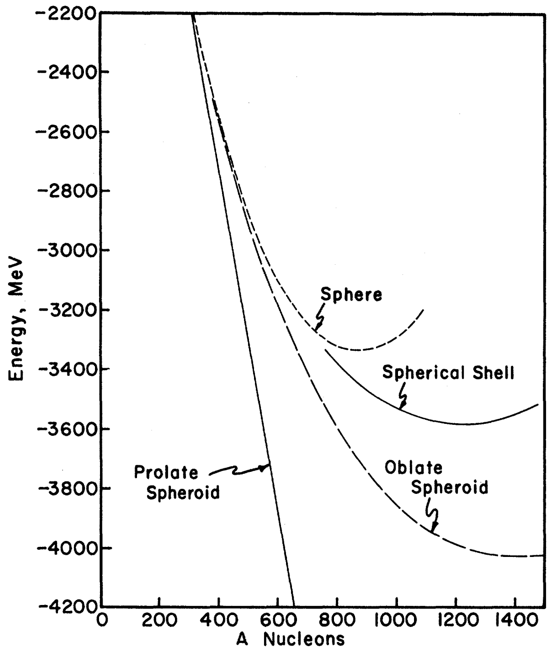

FIG. 1. Energies of nuclei with various configurations. For the spheroids and spherical shell, energies are for stable configuration.

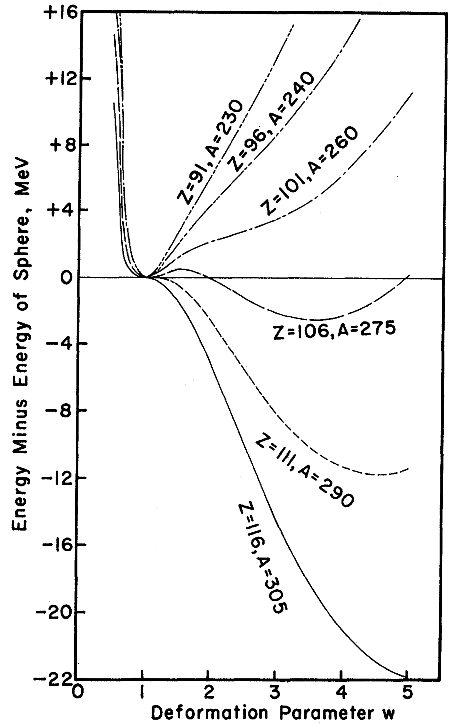

FIG. 2. Energies of spheroids with fixed A and Z as a function of deformation. w = (axis of rotation)/(rotated axis).

en by Eq. (1), is shown for comparison. All these are plotted with Green's values for f_1, f_2, and r_C. If r_C is reduced to 1.1 fm, all the curves are moved upwards somewhat, the oblate spheroid and the spherical shell moving somewhat more than the prolate spheroid, which moves only about 100 MeV. (The smaller value of r_C would probably be more appropriate for the shell, since r_C as found by Green includes an effect of lower proton density at the center of the nucleus.) Beta stability for the sphere, shell, and oblate spheroid are fairly similar, with Z/A ranging from 0.4 to 0.3 as A goes from 200 to 1000.

The most interesting case is evidently the prolate spheroid. It accepts additional nucleons much more readily than the other configurations. Its beta-stability curve is quite flat, with Z/A remaining at about 0.38 for A past 1500 nucleons. The prolate spheroid continues to bind the last nucleon by about 5 MeV even past $A = 3000$. Its minor axis (diameter) remains quite constant, being 8 fm for $A = 400$ and diminshing to 6 fm for $A = 1500$; it scarcely diminishes further by $A = 3000$.

More importantly, however, the prolate deformation begins to show an energetic advantage over a sphere when Z is greater than 104. (The atomic number at which deformation is favored is approximately proportional to the choice of r_C.) Figure 2 shows the total energy of A nucleons including Z protons, for several values of Z and A, as a function of w, the ratio of the axis of rotation to the rotated axis of the ellipse generating the spheroid. The values of Z chosen are those given by the beta-stability condition for the sphere, Eq. (2) with $g_4 = g_5 = 1$. We have also done calculations assuming 15 or 45 more neutrons for a given atomic number. The shape of the energy curves is almost entirely insensitive to the number of neutrons, depending chiefly on Z. Adding 15 neutrons makes the spherical shell slightly more favorable, with changes of the order of 0.1 MeV; adding 45 neutrons actually favors the prolate shape by about 0.5 MeV more than at beta stability.

The shape of the energy curve as a function of deformation was noted by Weiszäcker in 1939.[4] If a nucleus has more than some critical number of protons (found here to be 104), the spherical shape is only a shallow local minimum of energy, with a very elongated "stable" shape of considerably lower energy; for still higher Z, the spherical shape is not even a minimum. It may be seen from Fig. 2 that the minimum is always very shallow, and that the energy maximum encountered in a transition to the very elongated prolate shape is less than 1 MeV. Such nuclei must be extremely susceptible to spontaneous fission.[5]

In spite of the crudeness of the semiempirical formula, we believe that our predictions are fairly reliable. Deviations from that formula occur chiefly at closed shells, and it is generally agreed that around $Z=100$ we are about in the middle of filling a shell. In this case, the Nilsson levels of individual nucleons are quite close together, forming almost a continuum, so that the liquid-drop model should be a good approximation. Furthermore, collective interactions have the tendency to make the shape of nuclei prolate spheroids even in regions of Z and A where the liquid-drop model predicts spheres. We therefore believe that a liquid-drop prediction of spheroidal shape should be taken seriously.

These results will be modified by shell structure. This has an important influence on the energy only for the spherical shape. The next magic nucleus after Pb^{208} is expected to have 184 neutrons and, to keep the ratio of neutrons to protons close to equilibrium, probably 114 protons (which is not really a good closed shell).

The energy reduction for this semimagic nucleus is likely to be considerably less than for Pb^{208} for which it is about 15 MeV. On the other hand, without shell structure, the elongated shape has an energy about 18 MeV less than the sphere for $Z=114$. We therefore believe that shell effects are unlikely to make the nucleus $Z=114$ stable.

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*Work supported in part by the U. S. Office of Naval Research.

†National Science Foundation Predoctoral Fellow.

[1]H. Bethe and R. Bacher, Rev. Mod. Phys. 8, 165 (1936).

[2]A. Green, Rev. Mod. Phys. 30, 569 (1958).

[3]J. Nemeth, private communication.

[4]C. F. v. Weiszäcker, Naturwiss. 27, 133 (1939).

[5]N. Bohr and J. Wheeler, Phys. Rev. 56, 426 (1939).

FINE-STRUCTURE ANALYSIS OF ANALOG RESONANCES IN K^{41} †

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(Received 28 December 1966; revised manuscript received 9 January 1967)

This Letter reports an analysis of the fine structure observed in a high-resolution study of analog resonances in K^{41} using $Ar^{40}(p,p)Ar^{40}$.[1] The analysis uses a K-matrix theory of fine structure in nuclear reactions.[2] The S matrix is related to the K matrix by

$$S_{cc'}=\exp(i\delta_c)[(1-i\pi K)(1+i\pi K)^{-1}]_{cc'}\exp(i\delta_{c'}). \quad (1)$$

The K matrix is derived from a shell-model approach to reaction theory and has a resonant part

$$K_{cc'}{}^R=(2\pi)^{-1}\sum_\lambda \Gamma_{\lambda c}{}^{\frac{1}{2}}\Gamma_{\lambda c'}{}^{\frac{1}{2}}/(E-E_\lambda), \quad (2)$$

where $\Gamma_{\lambda c}=2\pi|\langle X_\lambda|V_e|c\rangle|^2$ is the partial width for decay of the state X_λ through channel c. The V_e is the shell-model effective interaction and X_λ is obtained by diagonalizing H_0+V_e on the set of discrete eigenstates of H_0. This H_0 is an independent-particle Hamiltonian with a Saxon-Woods potential, which is used to generate the nonresonant phase shifts appearing in Eq. (1).

The resonance energies E_λ and the widths $\Gamma_{\lambda c}$ can be treated as parameters which are determined by fitting the experimentally observed resonances. This analysis has been carried out on the Duke data. The spin, parity, resonance energy, proton width, and alpha widths of every fine-structure resonance in the range of $E_p=1.6432$ MeV to $E_p=2.6020$ MeV have been tabulated.[3] From this tabulation we have calculated the reduced widths $\gamma_\lambda{}^2=\Gamma_\lambda/2kRP$ which are shown in Fig. 1.

The model chosen to describe this resonance structure is the following. Strongly coupled to the incident proton channel is what one calls the analog state, which acts as a doorway state.