GREEN'S-FUNCTION DERIVATION OF THE LOW EQUATION FOR THE SCATTERING AMPLITUDE IN DILUTE MAGNETIC ALLOY SYSTEMS

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The relationship between the dispersion-theory and perturbation-theory approach to scattering in dilute magnetic alloys is established.

A current problem in the theory of dilute magnetic alloys is the specification of the low-temperature ground-state energy and transport properties of a system of free electrons interacting with a random distribution of impurities which possess localized moments. In particular, several authors[1-5] have suggested that below a "Kondo" temperature, $T_K \sim \epsilon_F \exp[-N/2J\rho(0)]$, the spin polarization of the electron gas compensates that of the impurity for antiferromagnetic s-d coupling. Four separate techniques have been applied to investigate this low-temperature regime: equations of motion,[2-4] variational Ansatz,[1,5] the "LSZ" version of dispersion theory,[6] and perturbation theory.[7,9] Unfortunately, the predictions of the various approaches in the low-temperature regime do not always agree among themselves. Some of the discrepancies can be understood from the point of view of the perturbation approach as due to the inherent nonuniqueness of the solutions imposed by the restriction, ab initio, to the domain of logarithmic accuracy.[10] The approximation of logarithmic accuracy suffices to describe the physical properties of the system for $T \gg T_K$. However, at temperatures $T \sim T_K$ this approximation is no longer meaningful. The analytic continuations of higher temperature results, which are valid to logarithmic accuracy, provide well-defined but nonunique descriptions of the low-temperature properties of the system.

The purpose of this Letter is the establishment of the relationship between the dispersion-theory and the perturbation-theory approaches in order to examine the uniqueness of the former approach for $T \lesssim T_K$.

The result presented in this note is the derivation of Suhl's Low equation for the "on-shell" scattering amplitude using a generalization of Abrikosov's[7] Green's-function techniques to incorporate finite temperature as well as nonlocal spin-dependent and spin-independent impurity potentials. Aided by the direct relation of this method to perturbation theory, we demonstrate that Suhl's Low equation itself is restricted to logarithmic accuracy when interpreted in terms of the contact s-d interaction model.

The interaction of the electrons with the magnetic impurities is described by the Hamiltonian

$$\mathcal{H}_{\rm int} = \frac{-1}{N}\sum_{\vec{p}\vec{p}'}\{J_1(\vec{p}'\vec{p})\delta_{\alpha\alpha'}\delta_{\beta\beta'} + J_2(\vec{p}'\vec{p})\sigma_{\alpha'\alpha}{}^{i} S_{\beta'\beta}{}^{i}\} \times a_{n\beta'}{}^{\dagger} a_{n\beta} c_{\vec{p}'\alpha'}{}^{\dagger} c_{\vec{p}\alpha} \exp[i(\vec{p}-\vec{p}')\cdot\vec{R}_n]. \quad (1)$$

In the above we have used Abrikosov's pseudofermion representation for the impurity spins. Here $a_{n\beta}{}^{\dagger}, a_{n\beta}$ denote the creation and annihilation operators for pseudofermions. Also, $\sigma_{\alpha'\alpha}$ are the matrix elements of the Pauli spin matrices, $\vec{R}_n$ is the nth impurity coordinate, $J_1(\vec{p}', \vec{p})$ is the spin-independent impurity potential, and $J_2(\vec{p}', \vec{p})$ is the effective s-d exchange integral. We use a summation convention on all repeated Zeeman indices (α, β) and Cartesian indices (i, j, k). We note that the spin-independent impurity potential is diagonal in the extended pseudofermion space. The pseudofermions introduce additional "spurious" spin states which are eliminated ("frozen out") by assigning a single particle energy λ to the pseudofermions and taking the limit $\lambda/T \to \infty$ prior to performing the momentum integrals.

We now introduce the renormalized "four-tailed" vertex function. This vertex corresponds to the sum of all possible graphs in which an electron line $\vec{p}, \epsilon, \alpha$ enters and $\vec{p}', \epsilon + \omega_1 - \omega_2, \alpha'$ leaves; and a pseudofermion line ω_1, β enters and ω_2, β' leaves. We represent this vertex function by

$$\langle\alpha\beta|\Gamma(\vec{p}, i\epsilon; i\omega_1|\vec{p}', i(\epsilon+\omega_1-\omega_2); i\omega_2)|\alpha'\beta'\rangle. \quad (2)$$

For a given order of perturbation theory consisting of n bare J_1 vertices and m bare J_2 vertices, there are $(m+n)!/n!$ topologically distinct diagrams corresponding to the independent arrangements of the spin-dependent ver-

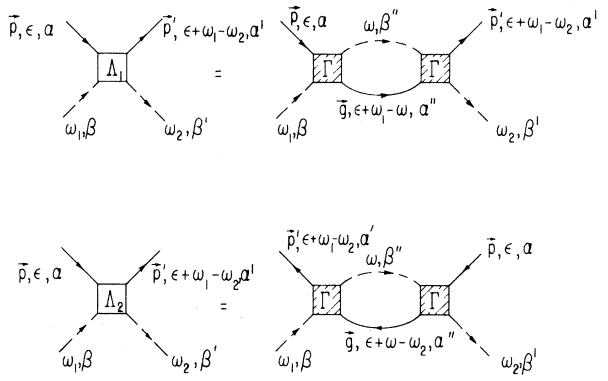

FIG. 1. Graphical equations for Λ_1 and Λ_2.

tices. We seek an integral equation which generates all of the distinct forms with each counted once and only once. Consider the integral equation

$$\Gamma=\Gamma_0+\Lambda_1+\Lambda_2. \tag{3}$$

Here Γ_0 is the bare vertex, $\Gamma_0=J_1+(\vec{\sigma}\cdot\vec{S})J_2$, and the functions Λ_1 and Λ_2 are generated by the integral equations in Fig. 1. The integral equation as it stands is incorrect for two reasons: First, the restriction to single-particle-hole intermediate states precludes ab initio the inclusion of any of the set of "nonparquet" graphs.[7] For a cut-off s-d model, the nonparquet graphs within each order of perturbation theory are of a lower order in the logarithmic divergence. A second, more important aspect of the integral equations we have written is that the topological forms generated are multiply counted in higher order iterations. This difficulty can be remedied, but the restoration of proper counting, which brings us directly to Suhl's Low equation, further imposes the restriction to logarithmic accuracy on the iteration solution.

We first write out the expressions for Λ_2 from Fig. 1 (the analysis for Λ_1 follows in an analogous fashion):

$$\langle\alpha\beta|\Lambda_2(\vec{p},i\epsilon;i\omega_1|\vec{p}',i(\epsilon+\omega_1-\omega_2);i\omega_2)|\alpha'\beta'\rangle$$
$$=T\sum_{\omega}\int\frac{d^3q}{(2\pi)^3}\frac{1}{i\omega-\lambda}\frac{1}{i(\epsilon+\omega-\omega_2)-\xi_{\vec{q}}}\langle\alpha\beta''|\Gamma(\vec{p},i\epsilon;i\omega|\vec{q},i(\epsilon+\omega-\omega_2);i\omega_2)|\alpha''\beta'\rangle$$
$$\times\langle\alpha''\beta|\Gamma(\vec{q};(\epsilon+\omega-\omega_2);i\omega_1|\vec{p}',i(\epsilon+\omega_1-\omega_2);i\omega)|\alpha'\beta''\rangle. \tag{4}$$

We first perform the sum over the Matsubara frequencies, $\omega=(2n+1)\pi T$. In the complex $z=i\omega$ plane there is a simple pole at $z=\lambda$, and a cut along $\mathrm{Im}[z+i(\epsilon-\omega_2)]=0$, which are contributed by the pseudofermion and electron Green's functions, respectively. In addition, there are series of overlapping cuts along the lines $\mathrm{Im}z=0$, $\mathrm{Im}(z-i\epsilon)=0$, and $\mathrm{Im}[z-i(\omega_1-\omega_2)]=0$ due to the vertex functions appearing in the integrand. This cut structure can be verified explicitly by an examination of the perturbation expansion of the vertex functions. The line integrals along the latter overlapping cuts are all weighted by an additional factor $e^{-\lambda/T}$ relative to the contribution from the electron propagator. Therefore, these contributions are reduced to zero when the spurious pseudofermion states are "frozen out" as $\lambda/T\to\infty$. Hence, in this limit, only the contributions from the electron propagator are of nonzero weight. The sum is now performed simply, reducing Eq. (4) to the form

$$\langle\alpha\beta|\Lambda_2(\vec{p},i\epsilon;i\omega_1|\vec{p}',i(\epsilon+\omega_1-\omega_2);i\omega_2)|\alpha'\beta'\rangle$$
$$=-\int\frac{d^3q}{(2\pi)^3}\frac{n(\xi_{\vec{q}})}{i\epsilon-\xi_{\vec{q}}}\langle\alpha\beta''|\Gamma(\vec{p},i\epsilon;i(\epsilon+\omega_2-\xi_{\vec{q}})|\vec{q},\xi_{\vec{q}};i\omega_2)|\alpha''\beta'\rangle$$
$$\times\langle\alpha''\beta|\Gamma(\vec{q},\xi_{\vec{q}};i\omega_1|\vec{p}',i(\epsilon+\omega_1-\omega_2);i(\epsilon+\omega_2-\xi_{\vec{q}}))|\alpha'\beta''\rangle. \tag{5}$$

The $n(\xi_{\vec{q}})$ in the above equation is the Fermi factor. The limiting case of the vertex function corresponding to the electron-impurity scattering amplitude is the retarded analytically continued form,

$$\Gamma^R(\vec{p},\epsilon|\vec{p}',\epsilon)=\lim_{\substack{i\omega_1,\,i\omega_2\to\lambda\to 0\\ i\epsilon\to\epsilon}}\Gamma(\vec{p},i\epsilon;i\omega_1|\vec{p}',i(\epsilon+\omega_1-\omega_2);i\omega_2). \tag{6}$$

In performing the analytic continuations one must take into account the overlapping cut structure in the complex ϵ plane,[8,11] which leads to the unitarity relations satisfied by $\Gamma(\vec{p}, \epsilon | \vec{p}', \epsilon)$. We obtain, from Eq. (5),

$$\langle\alpha\beta|\Lambda_2(\vec{p}, \epsilon | \vec{p}', \epsilon)|\alpha'\beta'\rangle = -\int \frac{d^3q}{(2\pi)^3} \frac{n(\xi_{\vec{q}})}{\epsilon - \xi_{\vec{q}} + i\delta} \langle\alpha\beta''|\Gamma(\vec{p}, \epsilon + i\delta; \epsilon - \xi_{\vec{q}} | \vec{q}, \xi_{\vec{q}}; 0)|\alpha''\beta'\rangle \times \langle\alpha''\beta|\Gamma(\vec{q}, \xi_{\vec{q}}; 0 | \vec{p}', \epsilon - i\delta; \epsilon - \xi_{\vec{q}})|\alpha'\beta''\rangle. \quad (7)$$

The equation as it stands still suffers from multiple counting of the topologically distinct vertex forms. As can be demonstrated explicitly from examples in third- and fourth-order perturbation theory[8] these counting inconsistencies can be remedied, but only to logarithmic accuracy, by placing the vertex functions appearing in the integrand of Eq. (7) on the energy shell of the intermediate hole propagator. This procedure reduces the equations for Λ_1 and Λ_2 to the forms

$$\langle\alpha\beta|\Lambda_1(\vec{p}, \epsilon | \vec{p}, \epsilon)|\alpha'\beta'\rangle = \int \frac{d^3q}{(2\pi)^3} \frac{n(-\xi_{\vec{q}})}{\epsilon - \xi_{\vec{q}} + i\delta} \langle\alpha\beta|\Gamma(\vec{p}, \xi_{\vec{q}} | \vec{q}, \xi_{\vec{q}})|\alpha''\beta''\rangle\langle\alpha''\beta''|\Gamma(\vec{p}', \xi_{\vec{q}} | \vec{q}, \xi_{\vec{q}})^* |\alpha'\beta'\rangle, \quad (8a)$$

$$\langle\alpha\beta|\Lambda_2(\vec{p}, \epsilon | \vec{p}, \epsilon)|\alpha'\beta'\rangle = -\int \frac{d^3q}{(2\pi)^3} \frac{n(\xi_{\vec{q}})}{\epsilon - \xi_{\vec{q}} + i\delta} \langle\alpha\beta''|\Gamma(\vec{p}, \xi_{\vec{q}} | \vec{q}, \xi_{\vec{q}})|\alpha''\beta'\rangle\langle\alpha''\beta|\Gamma(\vec{p}, \xi_{\vec{q}} | \vec{q}, \xi_{\vec{q}})^* |\alpha'\beta''\rangle. \quad (8b)$$

From rotational symmetry, Γ can be divided into a scalar and a vector form,

$$\Gamma = \Gamma^{(s)} + (\vec{\sigma}\cdot\vec{S})\Gamma^{(v)}. \quad (9)$$

The coupled nonlinear integral equations for the scalar and vector vertex functions are

$$\Gamma^{(s)}(\vec{p}, \epsilon | \vec{p}', \epsilon) = \frac{J_1(\vec{p}, \vec{p}')}{N} - \frac{1}{4\pi N} \int_{-\epsilon_F}^{\infty} \frac{\rho(\xi_{\vec{q}}) d\Omega_{\vec{p}\cdot\vec{q}} d\xi_{\vec{q}}}{\epsilon - \xi_{\vec{q}} + i\delta} \times \{\Gamma^{(s)}(\vec{p}, \xi_{\vec{q}}, | \vec{q}, \xi_{\vec{q}})\Gamma^{(s)}(\vec{p}', \xi_{\vec{q}} | \vec{q}, \xi_{\vec{q}})^* + S(S+1)\Gamma^{(v)}(\vec{p}, \xi_{\vec{q}} | \vec{q}, \xi_{\vec{q}})\Gamma^{(v)}(\vec{p}', \xi_{\vec{q}} | \vec{q}, \xi_{\vec{q}})^*\}; \quad (10a)$$

$$\Gamma^{(v)}(\vec{p}, \epsilon | \vec{p}, \epsilon) = \frac{J_2(\vec{p}, \vec{p}')}{N} - \frac{1}{4\pi N} \int_{-\epsilon_F}^{\infty} \frac{\rho(\xi_{\vec{q}}) d\Omega_{\vec{p}\cdot\vec{q}} d\xi_{\vec{q}}}{\epsilon - \xi_{\vec{q}} + i\delta} \{\Gamma^{(s)}(\vec{p}, \xi_{\vec{q}} | \vec{q}, \xi_{\vec{q}})\Gamma^{(v)}(\vec{p}', \xi_{\vec{q}} | \vec{q}, \xi_{\vec{q}})^* + \Gamma^{(v)}(\vec{p}, \xi_{\vec{q}} | \vec{q}, \xi_{\vec{q}})\Gamma^{(s)}(\vec{p}', \xi_{\vec{q}} | \vec{p}, \xi_{\vec{q}})^* - \tanh(\xi_{\vec{q}}/2T)\Gamma^{(v)}(\vec{p}, \xi_{\vec{q}})\Gamma^{(v)}(\vec{p}', \xi_{\vec{q}} | \vec{q}, \xi_{\vec{q}})^*\}. \quad (10b)$$

These coupled integral equations are manifestly identical to those derived by Suhl. It is important to note the consequence of the step in derivation which restored the proper counting, viz. putting the vertex functions in the integrand on the energy shell. This procedure has forced the vertex function to exhibit no singularities off the real ϵ axis. We emphasize that this procedure generates the perturbation theory at best to logarithmic accuracy in terms of a restricted s-d cutoff model. Therefore, from the point of view of perturbation theory, the absence of singularities of $\Gamma(\vec{p}, \epsilon | \vec{p}', \epsilon)$ in the upper half of the complex ϵ plane is a consequence of a variable substitution designed to compen-

sate within logarithmic accuracy for the incorrect counting of certain diagrams. We have demonstrated elsewhere[9] that a linear theory, yielding different analytic properties for $\Gamma(\vec{p}, \epsilon | \vec{p}', \epsilon)$, also generates the perturbation-theory results to logarithmic accuracy. Therefore, the failure of the requirement of logarithmic accuracy to specify uniquely the analytic structure of the vertex function is shown by construction of examples.

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[10]Consider a function $f(\epsilon)$ represented by a power series

$$f(\epsilon) = \sum_{n=0}^{\infty} \sum_{m=1}^{n} (J/\epsilon_F)^n a_{mn} \ln^{n-m}(\epsilon/\epsilon_F).$$

If $f(\epsilon)$ is specified to logarithmic accuracy, only the coefficients a_{n1} are known. Hence, there exist an infinite number of different functions, all equivalent within logarithmic accuracy, corresponding to the arbitrary choice of the a_{nm} for $m \neq 1$.
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HELICON MODES IN PURE TYPE-II SUPERCONDUCTORS IN THE HIGH-FIELD REGION*

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In a recent paper, Maxfield and Johnson[1] have reported an observation of helicon-like resonances in pure Nb in the mixed state. We would like to present some theoretical considerations related to this phenomenon. In a recent work[2] we have obtained the expression for the complex conductivity in a pure type-II superconductor in an external field H_0 close to the upper critical field H_{c2}. In particular, we study the low-frequency limit (i.e., $\omega \ll \pi T_{c0}$, where ω is the frequency of the oscillating field and T_{c0} is the transition temperature in the absence of field; note that we use the system of units where $\hbar = c = k_B = 1$). In that limit we show that the reactive part of the complex conductivity vanishes identically (up to order $H_{c2}-H$) in the geometry where the oscillating current flows in the plane perpendicular to the static external field (Fig. 1). This is exactly the geometry which is appropriate for the helicon experiment.

The electromagnetic response of the type-II superconductor can then be interpreted as the one of a normal metal in which the number of carriers N_n is a function of the static magnetic field as well as the temperature.

The relevant expression for the complex conductivity which appears in the expression for the current[2]

$$j_{\pm}(\vec{q}, \omega) = Q_{\pm}(\vec{q}, \omega) A_{\pm}(\vec{q}, \omega) \qquad (1)$$

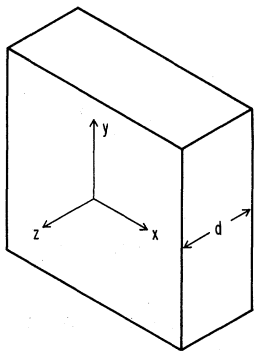

FIG. 1. The geometry of the helicon resonance experiment.