THERMODYNAMICS OF VORTEX FLOW IN SUPERCONDUCTORS

G. B. Yntema
United Aircraft Research Laboratories, East Hartford, Connecticut
(Received 26 August 1966; revised manuscript received 27 March 1967)

An expression is derived in this communication for the force which is applied to a straight vortex[1] line in a superconductor by the combination of electrical currents[2,3] and a thermal gradient.[4] We assume that the inertia associated with motion of the vortex may be neglected; so we identify force with a vector such that the energy dissipated by the motion is the scalar product of this vector and the displacement. The derivation is thermodynamic, since we are ignorant of the mechanisms through which the force is applied.

Consider a sample of type-II superconductor having translational symmetry in some direction. This symmetry applies not only to the surface of the sample but also to the distributions within the sample of temperature, of alloy composition, and of the density and effectiveness of pinning[5] sites for vortices. We further specify that the direction of the magnetic field outside the superconductor is everywhere that of the sample symmetry. Thus the sample may be within an infinitely long solenoid, which contributes magnetic field $\vec{H}_{\mathrm{sol}}$. Thus also, electrical currents may be borne by the superconductor provided that the current density, both within the superconductor and in the external circuits, possesses the translational symmetry and has no component in the direction of symmetry. Figure 1 shows the cross section of such a sample and external circuits.

The temperature distribution within the superconductor is presumed to be fixed by an artifice:

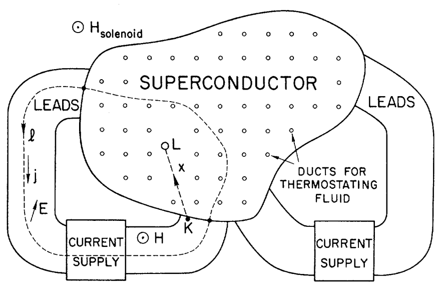

FIG. 1. Cross section of translationally symmetric specimen. A vortex is formed at K and moves into the hole at L. The dashed curve follows a flow line of electrical current in an attached circuit and is completed by an arbitrary contour within the superconductor.

Ducts run through the sample in the direction of symmetry. Each duct is maintained at its specified temperature by a fluid pumped along it.

We shall consider energy and entropy relations which apply to the following representative process.[6] A vortex parallel to the direction of symmetry is formed at the surface of the sample along some line K. Another vortex is discharged into a hole which runs in the direction of symmetry and is located at L. The cross-sectional area of the hole is sufficient so that we may neglect the quantum restriction on the magnetic flux trapped within it. But the hole is assumed to be small enough so that the differences in temperature between points on its surface are negligible. As still another part of the same process, we specify that vortices at positions between K and L move as follows. A surface composed of lines in the direction of symmetry is selected so that it connects K and L. The trace of this surface on a cross section of the sample is indicated in Fig. 1. Distance from K along the trace is denoted by x. The vortices which are less distant from this surface than about one-half the typical spacing between vortices form a sequential array which extends from K to L. We specify that each vortex in this array moves to the position originally occupied by its neighbor in the direction toward L. The net results of the whole process are a change in the strength of the supercurrent sheath on the surface of the hole, possibly a change in the heat content of the thermostating fluids in the ducts near the array of moving vortices, and possibly a change in the energy stored in the external sources of electrical energy.

We assume that the entropy produced within the sample is the sum of two terms. One of these, contributed by ordinary processes of thermal conduction, is independent[7] of the motion of vortices. The other is independent of ∇T and can be attributed to vortex motion. We denote by $\delta S/\delta x$ the entropy produced per unit length per unit distance traveled by a single vortex. The magnitude of this quantity depends on velocity and on attributes of the sample, including pinning.

In general, when a vortex is added to a superconductor of fixed temperature distribution, the density of vortices may be affected throughout the sample. But the process of sequential vortex progression described above involves no net change in number nor distribution of vortices. Accordingly, we assume that its thermal effects are confined to the vicinity of that surface which is followed by the sequential displacements. Though we have thus restricted the discussion to local effects, we must distinguish between (1) the heat which a superconductor absorbs locally when and where a vortex is formed and (2) the net heat absorbed in the region of formation if a vortex is both formed and then moved to another region. The latter heat, per unit length of vortex, we denote by Q_{nl}. (We assume that the formation and destruction of vortices are not dissipative[8] when performed at the rates typical of vortex-flow experiments.) A vortex moving within a superconductor of fixed temperature distribution may cause it to exchange heat with the thermostatting device. We denote by $\delta q/\delta x$ the heat thus absorbed per unit length per unit distance traveled.

The sequential vortex progression leaves the entropy of the sample unchanged:

$$0=\frac{Q_{nl}(K)}{T(K)}+\int_K^L dx\left[\frac{1}{T(x)}\frac{\delta q}{\delta x}+\frac{\delta S}{\delta x}\right]-\frac{Q_{nl}(L)}{T(L)}. \qquad (1)$$

We now consider another sample, similar to the first except that the hole into which a vortex is expelled is situated a bit farther from K so that a few more vortices are involved in the sequential progression. An equation similar to Eq. (1) applies to the second sample, and comparing it with Eq. (1) we obtain

$$0=(\delta q/\delta x)/T+\delta S/\delta x-dS_d/dx, \qquad (2)$$

where we have introduced $S_d\equiv Q_{nl}/T$ to represent the entropy per unit length delivered[4] by a vortex to the surface at which the vortex is destroyed. At a place where there is no hole nor surface, S_d is defined as the entropy per unit length which a vortex would deliver if a hole existed there.

Equation (2) indicates that a vortex may make a reversible contribution as well as a dissipative contribution to the heat in a region through which it passes. If vortices flow as an array, which has density n_v and velocity $\vec{v}_v$, the heat which they deposit reversibly[9] per unit time per unit volume within the superconductor is, from Eq. (2), $-Tn_v\vec{v}_v\cdot\nabla S_d$. This may also be written as $-(cT/\Phi_v)(\langle\vec{E}\rangle\times\langle\vec{B}\rangle/\langle B\rangle)\cdot\nabla S_d$, where c is the velocity of light, Φ_v is the fluxoid strength of each vortex, $\langle\vec{E}\rangle$ is the electric field smoothed over the distance of separation between vortices, and $\langle\vec{B}\rangle$ is magnetic field similarly smoothed. The second expression is obtained by use of the relation[10,11] $c\langle\vec{E}\rangle=-\vec{v}_v\times\langle\vec{B}\rangle$.

We denote by $\vec{B}_h$ the strength of magnetic induction within the hole at L when in equilibrium with the vortices in the surrounding superconductor. At a place where there is no hole, $\vec{B}_h$ is defined as the strength which would exist if there were a hole. The sequential vortex progression increases the energy of the combination of superconductor and magnetic field by the amount $\vec{B}_h(L)\cdot\vec{\Phi}_v/4\pi$. Thus

$$\vec{B}_h(L)\cdot\vec{\Phi}_v/4\pi =Q_{nl}(K)+\int_K^L dx\,(\delta q/\delta x)-Q_{nl}(L)+W_e, \qquad (3)$$

where W_e is the energy input per unit length from external electrical sources. It is easy to evaluate W_e if we specify that the impedances of the external circuits are such that the current distribution outside the sample is constant during the motion of the vortices. We calculate the energy supplied by the external circuit to each flow line of current. If the flow line passes through the superconductor, we select a contour within the superconductor to connect the point of entry with the point of exit. We need merely to take care that this contour lie deep within the superconductor and not intersect the surface which is swept out by the sequentially progressing vortices. In Fig. 1 the dashed loop indicates such an external flow line and associated internal contour. Distance along the loop is denoted by l. If we integrate on any loop over the time interval during which the vortex progression occurs, $\int dt\oint d\vec{l}\cdot\vec{E}$ is either $-\Phi_v/c$ or 0, depending on whether the hole at L is encircled by the loop. The instantaneous electric field is identically 0 within the superconductor on a contour such as we have specified. Therefore, $-\int dt\oint d\vec{l}\cdot\vec{E}$ is the energy supplied by the external circuit per unit of current on any flow line, whether the flow line is itself a loop or is completed as specified above. The current on those flow lines for which the loops encircle the hole at L is just the current which encircles a line lying outside the superconductor close to and parallel to the line K. It follows that

$$W_e = \vec{H}(K)\cdot\vec{\Phi}_v/4\pi, \tag{4}$$

where $\vec{H}(K)$ is the magnetic field just outside the superconductor at K.

Equations (3) and (4) together indicate that $\vec{B}_h$ just inside the superconductor is equal to $\vec{H}$ just outside. The boundary values of $\vec{B}_h$ are therefore specified by the external current distribution.

Again we consider a sample in which the hole is a bit farther from K. By comparing Eq. (3) with the similar equation for the second sample, we obtain

$$\frac{1}{4\pi}\frac{d}{dx}(\vec{B}_h\cdot\vec{\Phi}_v) = \frac{\delta q}{\delta x} - \frac{d}{dx}Q_{nl}. \tag{5}$$

Eliminating $\delta q/\delta x$ between Eqs. (2) and (5) yields

$$T\frac{\delta S}{\delta x} = -\frac{1}{4\pi}\frac{d}{dx}(\vec{B}_h\cdot\vec{\Phi}_v) - S_d\frac{dT}{dx}. \tag{6}$$

We assume that any kinetic energy associated with the motion of the vortices may be neglected. The energy dissipated per unit distance traveled must therefore be the component of force in the direction of travel. Since the direction of motion is subject to manipulation,[12,13] we generalize to

$$\vec{F} = -(1/4\pi)\nabla(\vec{B}_h\cdot\vec{\Phi}_v) - S_d\nabla T, \tag{7}$$

where $\vec{F}$ is the force on the vortex per unit length.

If a superconductor is maintained at a fixed uniform temperature, is in a constant applied field $\vec{H}(\vec{r})$, and has no electrical connections to external circuits, then it may be shown that the integral $\int d^3\vec{r}(\vec{B}-\vec{H})^2/8\pi$ taken over all space is the magnetic component of the free energy which is minimized at equilibrium. We therefore regard $(\vec{B}_h-\vec{H}_{\mathrm{sol}})\cdot\vec{\Phi}_v/4\pi$ as the "chemical potential" per unit length of a vortex, recalling that $\vec{B}_h$ is by definition in equilibrium with vortices in the superconducting material surrounding the hole. Equation (7) shows that in a state in which $\vec{F}=0$, a gradient of temperature supports a gradient of this "chemical potential" in proportion to the entropy of transport. In this respect a superconducting pipe with a radial temperature gradient in its wall is closely analogous to a permeable membrane between containers of gas at different temperatures.

We can now describe the formal framework of the relations which govern the flow of straight vortices. One equation obeyed by $\vec{v}_v$ and n_v (except at surfaces of the superconducting material) is that of continuity: $\partial n_v/\partial t = -\nabla\cdot(n_v\vec{v}_v)$. Another is the balance between dissipative drag and the force described by Eq. (7). Measurements[3] of drag indicate that it is the sum of a (viscous)[14] term proportional to $\vec{v}_v$ and a (pinning) term independent of $\vec{v}_v$. The quantities $\vec{B}_h$ and S_d, appearing in Eq. (7), each depend on n_v in measurable fashion. (The relation between $\vec{B}_h$ and $\langle\vec{B}\rangle$ is that between applied $\vec{H}$ and $\langle\vec{B}\rangle$ at equilibrium in a long, thin, electrically isolated isothermal sample parallel to $\vec{H}$.) The external current distribution determines the values of n_v at the surface of the superconductor.

The quantity $\vec{B}_h$, defined above, is regarded by some as the magnetic field inside the sample. But we shall show that the quantity so defined cannot be equivalent to any quantity $\vec{H}$ which is defined by geometric relationships with its sources, which are electric currents and magnetic materials. (Any $\vec{H}$ which appears in expressions for energy transferred magnetically must be definable in terms of sources.) Suppose that two long cylindrical rods with congruent cross sections are fabricated from type-II superconducting alloys and are placed parallel to a uniform applied $\vec{H}_{\mathrm{sol}}$ in a liquid-helium bath. Rod No. 1 is uniform in alloy composition but is heated internally (e.g., by gamma rays) so that a radial gradient of temperature is maintained in it. In the equilibrium (i.e., $\vec{F}=0$) configuration of vortices, $\vec{B}_h$ at the center of this rod is less than $\vec{H}_{\mathrm{sol}}$, according to Eq. (7). The smoothed field $\langle\vec{B}(\vec{r})\rangle$ within rod No. 1 is determined by the combination of alloy composition, $T(\vec{r})$, and $\vec{B}_h(\vec{r})$. Rod No. 2 is kept entirely at bath temperature. But we specify that its alloy composition is graduated in whatever way is required so that $\langle\vec{B}(\vec{r})\rangle$ everywhere in it, in the equilibrium configuration of vortices, is congruent to $\langle\vec{B}(\vec{r})\rangle$ in rod No. 1. The fields $\vec{H}(\vec{r})$ in the two rods are congruent[15] because all possible sources have been specified so as to be congruent. But $\vec{B}_h$ at the center of rod No. 2, being equal to $\vec{H}_{\mathrm{sol}}$, is different from $\vec{B}_h$ at the center of rod No. 1.

The boundary conditions on $\vec{B}_h$ at the surface of a superconductor are the same as those on $\vec{B}$ at the surface of a nonmagnetic normal conductor. This analogy may be exploited by defining $\vec{j}_h \equiv c\nabla\times B_h/4\pi$. The superconductor has no surface sheath of $\vec{j}_h$; and the normal component of $\vec{j}_h$ is continuous across the surface. In these respects $\vec{j}_h$ resembles the actual current in a normal conductor and differs from the actual smoothed current, $\langle\vec{j}\rangle \equiv c\nabla\times\langle\vec{B}\rangle/4\pi$, in a superconductor. The "flux-flow"-resistivity, Ettingshausen, and Nernst coefficients which have been reported[3,4] for type-II superconductors are ratios in which the denominators are current densities inferred from lead currents as though the samples were not superconducting. Such inferred current densities are a generalization of $\vec{j}_h$ applicable where flux lines are curved. In a translationally symmetric geometry,

$$-\nabla(\vec{B}_h\circ\vec{\Phi}_v)/4\pi=\vec{j}_h\times\vec{\Phi}_v/c. \qquad (8)$$

The author has benefited from discussions with F. A. Otter, Jr., and P. R. Solomon.

---

[1]A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. 32, 1442 (1957) [translation: Soviet Phys.—JETP 5, 1174 (1957)].

[2]Y. B. Kim, C. F. Hempstead, and A. R. Strnad, Phys. Rev. Letters 9, 306 (1962).

[3]A. R. Strnad, C. F. Hempstead, and Y. B. Kim, Phys. Rev. Letters 13, 794 (1964); Y. B. Kim, C. F. Hempstead, and A. R. Strnad, Phys. Rev. 139, A1163 (1965).

[4]F. A. Otter, Jr., and P. R. Solomon, Phys. Rev. Letters 16, 681 (1966).

[5]G. B. Yntema, in Proceedings of the Eighth International Conference on Low-Temperature Physics, London, 1962, edited by R. O. Davies (Butterworths Scientific Publications, Ltd., London, 1963), p. 119. P. W. Anderson subsequently suggested the same phenomenon [Phys. Rev. Letters 9, 309 (1962)].

[6]We do not suppose that this process actually occurs alone. A general flow of an array of vortices may be treated as the sum of several processes similar to this. The difference between a general flow and such a sum is that in the latter each vortex would follow a zigzag path. This difference is not important, since the zigzag feature of the representative process is not important in our analysis.

[7]For superconductors in which there is no pinning of vortices we may obtain (G. B. Yntema, to be published) an Onsager reciprocity relation for the flows of heat and of vorticity. Using this relation we can separate the rate of entropy production into a part independent of vortex motion and a part independent of ∇T. Thus encouraged, we hypothesize that ordinary thermal conduction processes are not perturbed by vortex motion even in samples in which pinning occurs.

[8]This assumption is a generalization from the observation [F. A. Otter, Jr., and G. B. Yntema, Bull. Am. Phys. Soc. 11, 107 (1966)] that in isothermal magnetization and demagnetization of well-annealed type-II samples both the magnetic moment and the absorbed heat are reversible over a broad range of $\vec{B}$.

[9]This is an analog of Peltier heating if $\nabla S_d \neq 0$ is produced by a gradient of composition; it is an analog of Thomson heating if $\nabla S_d \neq 0$ because of $\nabla T \neq 0$. It constitutes actual Peltier or Thomson heating if the Hall angle is nonzero so that $\vec{v}_v$ has a component parallel to $\vec{j}$. An expression for such Thomson heating has been derived by M. J. Stephen [Phys. Rev. Letters 16, 801 (1966)]. Such Peltier heating (with a gradient of magnetic induction maintaining $\nabla S_d \neq 0$) has been observed [A. T. Fiory and B. Serin, Phys. Rev. Letters 16, 308 (1966)].

[10]B. D. Josephson, Phys. Letters 16, 242 (1965).

[11]G. B. Yntema, Bull. Am. Phys. Soc. 10, 580 (1965); 11, 663(E) (1966). The monotonic change of phase of the order parameter during vortex flow relates flow rate uniquely to average electric field. The same description has subsequently been presented by P. W. Anderson [International Symposium on Quantum Fluids, Brighton, England, 1965, edited by D. Brewer (North-Holland Publishing Company, Amsterdam, 1966)] under the vivid label "phase slippage."

[12]Ivar Giaever, Phys. Rev. Letters 15, 825 (1965).

[13]P. R. Solomon, Phys. Rev. Letters 16, 59 (1966).

[14]M. J. Stephen and J. Bardeen, Phys. Rev. Letters 14, 112 (1965); J. Bardeen and M. J. Stephen, Phys. Rev. 140, A1197 (1965).

[15]The derivation by M. J. Stephen [Phys. Rev. Letters 16, 801 (1966)] of the force on a vortex line is based on the assertion that his function $G(T,H)$ is uniform when $\vec{F}=0$. This assertion cannot be true for both of the rods in our example [cf. Stephen's Eq. (3)]. We infer from the energy exchange expression in which Stephen's $\vec{H}$ is introduced that it is defined in terms of sources.