FACTORIZATION OF HELICITY AMPLITUDES AT HIGH ENERGIES*

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(Received 1 February 1967)

It has been known for some time that the factorization property of Regge-pole residues[1] leads to a sort of factorizability property at high energies in some experimental parameters of the scattering process. Thus Gribov and Pomeranchuk[2] were able to show that the nucleon-nucleon amplitude, at high energies, could be expressed as a product of $\pi+N\to\pi+N$ amplitudes, in the form

$$M=(A+\gamma^{(1)}\cdot Q_1B)(A+\gamma^{(2)}\cdot Q_2B).$$

A similar result in somewhat different form was deduced by Sharp and Wagner.[3]

We shall now show that these results generalize in a very simple way to arbitrary scattering processes involving particles of arbitrary mass and spin.

Let $f^{(s)}_{cd;ab}$, $f^{(s)}_{gh;ef}$, $f^{(s)}_{gd;eb}$, $f^{(s)}_{ch;af}$ be the s-channel helicity amplitudes[4] for the four processes

$$A+B\to C+D,$$
$$E+F\to G+H,$$
$$E+B\to G+D,$$
$$A+F\to C+H,$$

involving particles of arbitrary mass $m_A, m_B, \cdots$ and arbitrary spin $s_A, s_B, \cdots$. (Lower case subscripts refer, of course, to the helicities of the particles.) For each process let $f^{(s)}_{\nu\rho;\lambda\mu}$ be expressed as a sum of contributions arising from all the Regge poles R_n in the t channel, i.e.,

$$f^{(s)}_{\nu\rho;\lambda\mu}=\sum_n f^{(s)n}_{\nu\rho;\lambda\mu}, \tag{1}$$

where the sum on n is over all Regge poles. Then the following theorem holds: For each Regge pole R_n the <u>s-channel helicity amplitudes</u> <u>factorize as $s\to\infty$</u>, i.e.,

$$f^{(s)n}_{cd;ab}f^{(s)n}_{gh;ef}=f^{(s)n}_{gd;eb}f^{(s)n}_{ch;af} \text{ as } s\to\infty. \tag{2}$$

In the limit that only one Regge pole dominates, Eq. (2) implies the complete factorizability of the s-channel helicity amplitudes.

The result (2) follows quite trivially from the following observations:

(i) The residues of each Regge pole in the t channel satisfy the factorization theorem[1]

$$r_{c\bar{a};\bar{d}b}r_{g\bar{e};\bar{h}f}=r_{c\bar{a};\bar{h}f}r_{g\bar{e};\bar{d}b}, \tag{3}$$

where $\bar{a}, \bar{d}, \cdots$ refer to the helicities of $\bar{A}, \bar{D}, \cdots$, the antiparticles of $A, D, \cdots$, etc.

(ii) The crossing matrix[5] M is built up of a product of functions, one for each particle, and can therefore be factorized. For example,

$$f^{(s)}_{cd;ab}=\sum_{\substack{c',\bar{a}'\\ \bar{d}',b'}}M^{c'\bar{a}';\bar{d}'b'}_{ca;db}f^{(t)}_{c'\bar{a}';\bar{d}'b'}, \tag{4}$$

with

$$M^{c'\bar{a}';\bar{d}'b'}_{ca;db}=d^{s_c}_{c',c}(\chi_c)d^{s_a}_{\bar{a}',a}(\chi_a)d^{s_d}_{\bar{d}',d}(\chi_d)d^{s_b}_{b',b}(\chi_b)=M^{c'\bar{a}'}_{ca}M^{\bar{d}'b'}_{db}. \tag{5}$$

(iii) The rotation functions $d_{\lambda\mu}{}^J(z)$ possess the remarkable property that their leading term factorizes:

$$d_{\lambda,-\mu}{}^J(z)=\mathcal{a}_\lambda{}^J(z)\mathcal{a}_\mu{}^J(z) \text{ as } z\to\infty, \tag{6}$$

with

$$\mathcal{a}_\lambda{}^J(z)=\left[(-1)^\lambda\left(\frac{z}{2}\right)^J\frac{(2J)!}{(J+|\lambda|)!(J-|\lambda|)!}\right]^{1/2}. \tag{7}$$

The proof of (2) now reduces to simple algebraic manipulation. Each t-channel helicity amplitude is replaced by its Reggeized expression,[6] e.g.,

$$f^{(t)}_{c'\bar{a}';\bar{d}'b'}=(-1)^{1+v+\Lambda_{\bar{d}'b'}}\frac{\pi^2\sqrt{t}}{4(p_{\bar{D}B}p_{C\bar{A}})^{1/2}}[(1+\delta_{\bar{d}',b'})(1+\delta_{\bar{d}',-b'})(1+\delta_{c',\bar{a}'})(1+\delta_{c',-\bar{a}'})]^{1/2}$$
$$\times\sum_n(2\alpha_n+1)\zeta_n r^{(n)}_{c'\bar{a}';\bar{d}'b'}d^{\alpha_n}_{\Lambda_{\bar{d}'b'},-\Lambda_{c'\bar{a}'}}(-z_t), \tag{8}$$

where ζ_n is the usual signature factor

$$\zeta_n = \frac{1+\tau_n \exp[-i\pi(\alpha_n - v)]}{2\sin\pi(\alpha_n - v)},$$

$v = 0$ for mesonic poles,

$v = \frac{1}{2}$ for fermionic poles,

and

$$\Lambda_{c'\bar{a}'} = c' - \bar{a}', \text{ etc.},$$

and is then substituted into the right-hand side of Eq. (2). Using (4)-(6), the expression is easily regrouped into a form recognizable as the left-hand side of (2), which completes the proof.

Equation (2) suggests an interesting method for studying the behavior of the t-channel partial-wave helicity amplitudes at the thresholds $t = (m_A \pm m_C)^2$, $(m_B \pm m_D)^2$.

Combining (5) and (6) we can write, as $s \to \infty$,

$$f^{(s)}_{cd;ab} = \sum_n s^{\alpha_n} \zeta_n G^{(n)}_{ca;db}(t), \tag{9}$$

where G is a linear combination of the residues, which is independent of s as $s \to \infty$. The functions $G_{ca;db}$ share with $f^{(s)}_{cd;ab}$ the property of being free from kinematic singularities at $t = (m_A \pm m_C)^2$, $(m_B \pm m_D)^2$. Moreover, this remains true if in G the residues are replaced by the partial-wave amplitudes T^J for physical J. Thus the threshold behavior of the T^J is expressed by the analyticity of the linear combinations $G_{ca;db}$ of them.[7]

It is also worth noting that Eq. (2) yields some information about the behavior of the helicity amplitudes $f^{(s)}_{cd;ab}$ on the s-physical region boundary, which is $t = 0$ in the limit $s \to \infty$. The behavior,

$$f^{(s)}_{cd;ab} \propto (-t)^{\frac{1}{2}|\Lambda_{ab} - \Lambda_{cd}|} \tag{10}$$

which is implied by the conservation of angular momentum, is modified by (2) so that

$$f^{(s)}_{cd;ab} \propto (-t)^{\frac{1}{2}(|\Lambda_{ab}| + |\Lambda_{cd}|)} \tag{11}$$

a result which has extraordinary experimental repercussions.[8] An obvious application is to the behavior as $t \to 0$ of the density matrix elements for the decay of resonances such as ρ, $K^*(890)$ and $N^*(1236)$. An interesting discussion of some aspects of this problem has already been given.[9]

One of the authors (E.L.) would like to thank Dr. A. B. Lidiard for his hospitality at the Atomic Energy Research Establishment, Harwell.

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*Work supported in part by the U. S. Air Force Office of Scientific Research, OAR, under grant No. AF EOAR 65-36, with the European Office of Aerospace Research, U. S. Air Force.

[1]M. Gell-Mann, Phys. Rev. Letters 8, 263 (1962); V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters 8, 343 (1962).

[2]V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters 8, 412 (1962).

[3]D. H. Sharp and W. G. Wagner, Phys. Rev. 131, 2226 (1963).

[4]Our helicity amplitudes are related to those of M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) 7, 404 (1959), by

$$f^{(s)}_{cd;ab} = 2\pi\left(s\frac{p_{AB}}{p_{CD}}\right)^{1/2} f^{\text{J.W.}}_{cd;ab},$$

where, for instance, p_{AB} is the momentum of particles A and B in the s-channel center-of-mass system.

[5]T. L. Trueman and G. C. Wick, Ann. Phys. (N.Y.) 26, 322 (1964).

[6]M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. 133, B145 (1964). Our residues are the unmodified residues of the partial-wave helicity amplitudes $\langle c'\bar{a}'|T^J|\bar{d}'b'\rangle$ of definite parity (and symmetry, if it applies) in which all states are correctly normalized to unity.

[7]In the more usual method the arguments involving orbital angular momentum apply only to the top thresholds $t = (m_A + m_C)^2$ and $t = (m_B + m_D)^2$. This is elaborated by J. Franklin, Phys. Rev. 152, 1437 (1966), but the discussion of the kinematic behavior at the lower thresholds given there is incorrect.

[8]These will be discussed in a forthcoming publication.

[9]A. B. Kaidalov and B. M. Karnakov, Yadern. Fiz. 3, 814 (1966) [translation: Soviet J. Nucl. Phys. 3, 814 (1966)].