CLASSICAL AND QUANTUM SYSTEMS WITH TIME-DEPENDENT HARMONIC-OSCILLATOR-TYPE HAMILTONIANS*

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The unifying feature of the physical systems to be discussed in this Letter is the existence of a Hamiltonian of the form

$$H=(1/2\epsilon)[p^2+\Omega^2(t)q^2], \tag{1}$$

where q is a canonical coordinate, p is its conjugate momentum, $\Omega(t)$ is an arbitrary continuous function of t, and ϵ is a positive, real parameter. The original motivation for considering such systems was the desire to investigate the nature of the magnetic-moment series for a charged particle moving nonrelativistically in the relatively simple electromagnetic field for which the scalar potential is 0 and the vector potential is

$$\vec{A}=\tfrac{1}{2}h(t)\vec{B}_0\times\vec{r},$$

where $h(t)$ is a function of time, $\vec{B}_0$ is a constant vector, and $\vec{r}$ is the position vector; with this vector potential the magnetic field, $\vec{B}(t)$, is given by $h(t)\vec{B}_0$. The equations of motion for such a particle can be reduced to the equations corresponding to the Hamiltonian of Eq. (1) exactly,[1] in which case $\Omega(t)$ is $\frac{1}{2}B(t)$, and ϵ is the ratio of mass to charge.

If Ω is real, as it is in the charged-particle problem, the classical system whose Hamiltonian is given by Eq. (1) becomes oscillatory with arbitrarily large frequency as ϵ tends to 0. Corresponding to this fact, there exists an asymptotic series in positive powers of ϵ, the partial sums of which are adiabatic invariants of the system; the leading term of the series is $\epsilon H/\Omega$. In the charged-particle problem this adiabatic invariant is the magnetic-moment series. The results reported in this Letter stem from an application of the asymptotic theory of Kruskal[2] to the classical system represented by Eq. (1) with real Ω. It has been possible to apply Kruskal's theory to this system in closed form, and, as a consequence, to derive an exact invariant, a special case of which is the adiabatic invariant just mentioned. Although Ω was originally assumed to be real, the final results are valid for Ω complex. Also, the exact invariant is a constant of the motion of the quantum system whose Hamiltonian is given by Eq. (1).

First consider the classical system and take Ω to be real. In order to apply Kruskal's theory it is necessary to write the equations of motion as a first-order autonomous system such that all of the solutions are periodic in the independent variable in the limit $\epsilon=0$. This can be achieved by introducing a new independent variable s, defined by $s=t/\epsilon$, and treating t formally as a dependent variable. The system of equations thus obtained is

$$\begin{aligned} dq/ds&=p,\\ dp/ds&=-\Omega^2(t)q,\\ dt/ds&=\epsilon. \end{aligned} \tag{2}$$

Because t is now a dependent variable, this system is autonomous. In the limit $\epsilon=0$, the solution of the last equation is $t=$ constant and, therefore, the other two equations are just the harmonic-oscillator equations with a constant frequency. Since Ω is real, the dependent variables are all periodic in s with period $2\pi/\Omega(t)$ in the limit $\epsilon=0$, and the system of equations is in the form required by Kruskal's asymptotic theory. The details of the application of the theory will be given in a longer publication elsewhere. Here we limit ourselves to a brief summary of the method.

A central feature of the Kruskal theory is a transformation from the variables (q,p,t) to so-called "nice variables" which we may call (z_1,z_2) and φ. The nice variables are so chosen that a two-parameter family of closed curves in (q,p,t) space can be defined by the conditions $z_1=$ constant and $z_2=$ constant. These closed curves are called rings. The variable φ is an angle variable which we define in such a way that it changes by 2π if any ring is traversed once. The rings have the important property that the family of rings is mapped into itself if each ring is allowed to change with s according to Eqs. (2). In the general theory the transformation from the variables (q,p,t) to the variables (z_1,z_2) and φ is defined as an asymptotic series in positive powers of ϵ, and a prescription is given for determining the transformation order by order. However, in this example it has proven possible to obtain the transformation in explicit closed form in terms of the variables q and p and a function $\rho(t)$. It is also possible to invert the transformation explicitly.

For this problem it turns out that the rings lie in planes given by $t =$ constant. Therefore, we may use the rings to define an exact invariant, I, as the action integral

$$I = \oint_{\text{ring}} p dq. \tag{3}$$

Carrying out the integration explicitly by expressing I as an integral from 0 to 2π over the variable φ, we obtain

$$I = \tfrac{1}{2}[\rho^{-2}q^2 + (\rho p - \epsilon\rho' q)^2], \tag{4}$$

where ρ is a function of t satisfying

$$\epsilon^2\rho'' + \Omega^2\rho - \rho^{-3} = 0, \tag{5}$$

and the prime denotes differentiation with respect to t. The function ρ can be taken as any particular solution of Eq. (5). Although Ω was assumed to be real in the derivation, the quantity I is an invariant even if Ω is complex. It is easy to verify $dI/dt = 0$ for the general case of Ω complex by differentiating Eq. (4), using Eqs. (2) to eliminate dq/dt and dp/dt, and using Eq. (5) to eliminate ρ''.

It may appear that the problem of solving the linear system of Eqs. (2) has merely been replaced by the problem of solving the nonlinear Eq. (5). However this is not the case. Firstly, any one particular solution of Eq. (5) can be used in the formula for I with all initial conditions for Eqs. (2). For numerical work it is of practical importance that only one particular solution for ρ need be found. Secondly, we now have an exact invariant whose dependence on the dynamical variables, q and p, is explicit and simple. Thirdly, by virtue of the fact that ϵ^2 multiplies ρ'' in Eq. (5), it is straightforward to obtain a particular solution for ρ as a series in positive powers of ϵ^2. If Ω is real and the leading term in the series is taken to be $\Omega^{-1/2}$, then the series solution corresponds to the usual adiabatic-invariant series. It is interesting to speculate whether it is more useful in practice to calculate I with the truncated series solution for ρ than with the corresponding series expression for I truncated at the same power of ϵ. Fourthly, we can also solve Eq. (5) to obtain ρ as a power series in $1/\epsilon^2$ in terms of integrals. Finally, with the result expressed by Eqs. (4) and (5), it is possible to understand the nature of the adiabatic invariant more fully. Some progress along this line is contained in the following general discussion of I and ρ.

To within a constant factor, the invariant I given by Eq. (4) is the most general invariant of the linear system whose Hamiltonian is given by Eq. (1) that is a homogeneous quadratic form in q and p. This can be seen by writing the most general such invariant in terms of two linearly independent solutions, $f(t)$ and $g(t)$, of the linear system. If we generalize I to

$$I = E^2[\rho^{-2}q^2 + (\rho p - \epsilon\rho' q)^2], \tag{4$'$}$$

where E is an arbitrary constant, and compare it with the general quadratic invariant expressed in terms of f and g, then we can deduce that the two invariants are identical if ρ is given by

$$\rho = \gamma_1(\epsilon\alpha)^{-1}\left\{\frac{A^2}{E^2}g^2 + \frac{B^2}{E^2}f^2 + 2\gamma_2\left[\frac{A^2B^2}{E^4} - (\epsilon\alpha)^2\right]^{1/2} fg\right\}^{1/2}, \tag{6}$$

where A and B are arbitrary constants and the constants α, γ_1, and γ_2 are defined by

$$\alpha = fg' - gf', \quad \gamma_1 = \pm 1, \quad \gamma_2 = \pm 1. \tag{7}$$

Because there are two arbitrary constants in this formula for ρ, it is the general solution of Eq. (5) expressed in terms of f and g. Using this formula we can construct ρ explicitly for any Ω for which Eqs. (2) can be solved exactly. By constructing ρ in this manner for special cases, we can deduce that the series expansion of ρ in positive powers of ϵ^2 is at least sometimes convergent. For example, if $\Omega = bt^{-2n/(2n+1)}$, where b is a constant and n is any integer, the series expansion is a polynomial in ϵ^2, and therefore it is convergent with an infinite radius of convergence.

Having found the explicit form of the invariant defined by Kruskal, it is then possible to define a canonical transformation in which the new momentum is the invariant. Furthermore, a generating function for this transformation can also be found. If we denote the new coordinate by Q, its conjugate momentum by P, and the generating function by F, then the results can be written as

$$Q=-\tan^{-1}[\rho^2 p/q-\epsilon\rho\rho'],$$
$$P=\tfrac{1}{2}[\rho^{-2}q^2+(\rho p-\epsilon\rho' q)^2],$$
$$F=\tfrac{1}{2}\epsilon\rho^{-1}\rho' q^2\pm\rho^{-1}q(2P-\rho^{-2}q^2)^{1/2}$$
$$\pm P\sin^{-1}[\rho^{-1}q/(2P)^{1/2}]+(n+\tfrac{1}{2})\pi P$$
$$\left(-\frac{\pi}{2}\leq\sin^{-1}[\rho^{-1}q/(2P)^{1/2}]\leq\frac{\pi}{2},\ n=\text{integer}\right),$$
$$p=\partial F/\partial q,\quad Q=\partial F/\partial P,$$
$$\text{new Hamiltonian}=H+\frac{\partial F}{\partial t}=\frac{1}{\epsilon}\rho^{-2}P. \tag{8}$$

In the expression for F the upper or lower signs are taken according as $p-\epsilon\rho^{-1}\rho' q$ is greater than or less than 0. It is seen that Q is a cyclic variable in the new Hamiltonian, as it must be since P is an exact invariant.

Before leaving the classical theory, we note that the second-order differential equation for q, $\epsilon^2 d^2q/dt^2+\Omega^2(t)q=0$, is of the same form as the time-independent, one-dimensional Schrödinger equation if we let t represent the spatial coordinate and q represent the wave function. For bound states Ω is imaginary, and for continuum states Ω is real. Thus, the invariant I is a relationship between the wave function and its first derivative.

Let us now consider the quantum system whose Hamiltonian is given by Eq. (1), where q and p are now required to satisfy the commutation relation

$$[q,p]=i\hbar. \tag{9}$$

We also take ρ to be real, which is possible if Ω^2 is real. Using the commutation relation and the equation for ρ, it is easy to show that the quantity I which is an invariant of the classical system is also a quantum mechanical constant of the motion. That is, I satisfies

$$\frac{dI}{dt}=\frac{\partial I}{\partial t}+\frac{1}{i\hbar}[I,H]=0. \tag{10}$$

Therefore, I has eigenstates whose eigenvalues are time independent. These eigenstates and eigenvalues of I can be found by a method that is completely analogous to the method introduced by Dirac[3] for finding the eigenstates and eigenvalues of the Hamiltonian of a harmonic oscillator. We first introduce "raising" and "lowering" operators, $a^\dagger$ and a, that are defined by

$$a^\dagger=(1/\sqrt{2})[\rho^{-1}q-i(\rho p-\epsilon\rho' q)],$$
$$a=(1/\sqrt{2})[\rho^{-1}q+i(\rho p-\epsilon\rho' q)]. \tag{11}$$

These operators satisfy the relations

$$[a,a^\dagger]=\hbar,$$
$$aa^\dagger=I+\tfrac{1}{2}\hbar. \tag{12}$$

The operator a operating on an eigenstate of I produces an eigenstate of I whose eigenvalue is $\hbar$ lower than that of the original eigenstate. Similarly, $a^\dagger$ acting on an eigenstate of I increases the eigenvalue by $\hbar$. Once these properties are established, the normalizability of the eigenstates of I can be used to demonstrate that the eigenvalues of I are $(n+\frac{1}{2})\hbar$, where n is 0 or a positive integer. Letting $|n\rangle$ denote the normalized eigenstate of I whose eigenvalue is $(n+\frac{1}{2})\hbar$, we can express the relation between $|n+1\rangle$ and $|n\rangle$ as

$$|n+1\rangle=[(n+1)\hbar]^{-1/2}a^\dagger|n\rangle. \tag{13}$$

The condition which determines the state whose eigenvalue is $\frac{1}{2}\hbar$ is

$$a|0\rangle=0. \tag{14}$$

With these results the expectation value of the Hamiltonian in a state $|n\rangle$ can be calculated. The result is

$$\langle n|H|n\rangle=(1/2\epsilon)(\rho^{-2}+\Omega^2\rho^2+\epsilon^2\rho'^2)(n+\tfrac{1}{2})\hbar. \tag{15}$$

It is interesting to note that the expectation values of H are equally spaced at every instant and that the lowest value is always obtained with $n=0$, just as with the harmonic oscillator.

The quantum-mechanical results reduce to the usual ones for a harmonic oscillator if we take Ω to be real and constant and take $\rho=\Omega^{-1/2}$, so that $I=\epsilon H/\Omega$.

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[1]See, for example, F. Hertweck and A. Schlüter, Z. Naturforsch, 12a, 844 (1957).

[2]M. Kruskal, J. Math. Phys. 3, 806 (1962).

[3]P. A. M. Dirac, The Principles of Quantum Mechanics (Clarendon Press, Oxford, 1947), 3rd. ed.