PRECISE RELATIONS BETWEEN THE SPECTRA OF VECTOR AND AXIAL-VECTOR MESONS

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Two sum rules are derived, relating moments of the spectral functions of the vector and axial-vector currents. If it is assumed that the ρ and $A1$ mesons dominate these moments, then their masses must be in the ratio $m_{A1}/m_\rho=\sqrt{2}$, in very good agreement with experiment.

If chiral $SU(2)\otimes SU(2)$ were an exact symmetry of the ordinary sort, we should expect the ρ meson to be accompanied with an $I=1$ axial-vector meson of the same mass. This is certainly not the case; the best candidate for the role of chiral partner of the ρ is the $A1$, which has $m_{A1}{}^2\simeq 2m_\rho{}^2$. However, the recent successes of current algebra show that nature does obey some sort of chiral symmetry, manifested in the conservation or partial conservation of currents, and in their commutation relations. The question thus arises: What relations are imposed by current algebra upon the spectra of the 1^+ and 1^- mesons?

Our answer is contained in the following theorem: Assume that the vector and axial-vector currents obey the usual commutation relations,[1] with Schwinger terms[2] which are either c numbers or, if operators, contain no $\Delta I=1$ terms. Neglect the pion mass altogether, so that the axial vector as well as the vector currents are conserved.[3] Then

$$\int_0^\infty[\rho_V(\mu^2)-\rho_A(\mu^2)]\mu^{-2}d\mu^2=F_\pi{}^2, \tag{1}$$

where F_π is the usual pion-decay amplitude, and $\rho_{V,A}(\mu^2)$ are the spectral functions of the vector and axial-vector currents, defined by the formulas[4,5]

$$\langle V_a{}^\mu(x)V_b{}^\nu(0)\rangle_0=(2\pi)^{-3}\delta_{ab}\int d^4p\,\theta(p^0)e^{ip\cdot x}\rho_V(-p^2)[g^{\mu\nu}-p^\mu p^\nu/p^2], \tag{2}$$

$$\langle A_a{}^\mu(x)A_b{}^\nu(0)\rangle_0=(2\pi)^{-3}\delta_{ab}\int d^4p\,\theta(p^0)e^{ip\cdot x}\{\rho_A(-p^2)[g^{\mu\nu}-p^\mu p^\nu/p^2]+F_\pi{}^2\delta(p^2)p^\mu p^\nu\}. \tag{3}$$

If we further assume a very weak form of vector- and axial-vector-meson dominance, i.e., that matrix elements of the currents act at high momenta as if the currents were free $1^\pm$ fields,[6] then we also have

$$\int_0^\infty[\rho_V(\mu^2)-\rho_A(\mu^2)]d\mu^2=0. \tag{4}$$

Before proving these theorems, let us note some of their implications. The spectral functions $\rho_{V,A}(\mu^2)$ are measurable, in principle, from the cross sections for hadron production in electron-neutrino collisions. For the present, we can estimate $\rho_V(\mu^2)$ by using the hypothesis of ρ dominance:

$$\rho_V(\mu^2)\simeq g_\rho{}^2\delta(\mu^2-m_\rho{}^2). \tag{5}$$

Eqs. (1) and (4) now read

$$\int_0^\infty\rho_A(\mu^2)\mu^{-2}d\mu^2\simeq g_\rho{}^2m_\rho{}^{-2}-F_\pi{}^2, \tag{6}$$

$$\int_0^\infty\rho_A(\mu^2)d\mu^2\simeq g_\rho{}^2. \tag{7}$$

Hence, if $\rho_A(\mu^2)$ is sharply peaked about a point $\mu=m_A$, we must have[7]

$$m_A/m_\rho\simeq[1-F_\pi{}^2m_\rho{}^2/g_\rho{}^2]^{-1/2}. \tag{8}$$

Using ρ dominance and either current algebra[8] or the observed ρ width, we have $g_\rho{}^2\simeq 2F_\pi{}^2m_\rho{}^2$, so Eq. (8) gives

$$m_A/m_\rho\simeq\sqrt{2} \tag{9}$$

in extraordinary agreement with the observed[9] masses of the ρ and $A1$, for which $m_{A1}/m_\rho=1.41\pm0.01$.

Now to the proof of Eqs. (1) and (4). Define a three-point function

$$-i\epsilon_{abc}M^{\mu\nu\lambda}(q,p) \equiv \int d^4x d^4y \langle T\{A_a{}^\mu(x),\ A_b{}^\nu(y),\ V_c{}^\lambda(0)\}\rangle_0 \exp[-iq\cdot x - ip\cdot y]. \tag{10}$$

Our assumptions lead immediately to the following Ward identities:

$$\tfrac{1}{2}q_\mu M^{\mu\nu\lambda}(q,p) = \Delta_V{}^{\nu\lambda}(q+p) - \Delta_A{}^{\nu\lambda}(p), \tag{11}$$

$$\tfrac{1}{2}(q+p)_\lambda M^{\mu\nu\lambda}(q,p) = \Delta_A{}^{\mu\nu}(q) - \Delta_A{}^{\mu\nu}(p), \tag{12}$$

where Δ_V and Δ_A are the propagators of the vector and axial-vector currents. Multiply (11) with $(q+p)_\lambda$ and (12) with q_μ, and subtract; this gives

$$(q+p)_\lambda \Delta_V{}^{\nu\lambda}(q+p) = q_\lambda \Delta_A{}^{\nu\lambda}(q) + p_\lambda \Delta_A{}^{\nu\lambda}(p). \tag{13}$$

Equation (13) holds for all values of q_μ and p_μ, so each term must be the same linear function of its argument, i.e.,

$$K_\lambda \Delta_V{}^{\nu\lambda}(K) = K_\lambda \Delta_A{}^{\nu\lambda}(K) = C^{\nu\lambda}K_\lambda \tag{14}$$

with C a constant. Writing this in coordinate space and using current conservation, Eq. (14) becomes

$$\delta(x^0)\langle[V_a{}^0(x),\ V_b{}^\nu(0)]\rangle_0 = \delta(x^0)\langle[A_a{}^0(x),\ A_b{}^\nu(0)]\rangle_0 = -i\delta_{ab}C^{\nu\lambda}\partial_\lambda\delta^4(x), \tag{15}$$

so our theorem states the equality of the vector and axial-vector Schwinger terms! The vacuum expectation values in (15) can be evaluated from (2) and (3), with the result that they vanish for $\nu=0$, while for $\nu=1,2,3$ they are[10]

$$\delta(x^0)\langle[V_a{}^0(x),\ V_b{}^\nu(0)]\rangle_0 = -i\partial^\nu\delta^4(x)\int_0^\infty \rho_V(\mu^2)\mu^{-2}d\mu^2, \tag{16}$$

$$\delta(x^0)\langle[A_a{}^0(x),\ A_b{}^\nu(0)]\rangle_0 = -i\partial^\nu\delta^4(x)[F_\pi{}^2 + \int_0^\infty \rho_A(\mu^2)\mu^{-2}d\mu^2]. \tag{17}$$

Equation (1) now follows from (15)-(17).

In order to prove Eq. (4) we return to Eq. (11) and now set $q^\mu=0$. The left-hand side has a one-pion pole, which is all that survives at $q^\mu=0$, so Eq. (11) now reads

$$\tfrac{1}{2}iF_\pi\int d^4x\langle\pi_a|T\{A_b{}^\nu(y),\ V_C{}^\lambda(0)\}|0\rangle e^{-ip\cdot y} = \Delta_V{}^{\nu\lambda}(p) - \Delta_A{}^{\nu\lambda}(p), \tag{18}$$

where $|\pi_a\rangle$ is a covariantly normalized state representing a pion of zero energy and isospin index a. Next let $p^2\to\infty$. Our assumption[6] that the currents behave at infinity like free fields tells us that the coefficient of $g^{\nu\lambda}$ on the left-hand side behaves like $(p^2)^{-2}$, while the coefficient of $g^{\nu\lambda}$ on the right-hand side approaches

$$\frac{1}{p^2}\int_0^\infty[\rho_V(\mu^2)-\rho_A(\mu^2)]d\mu^2,$$

so Eq. (4) is necessary for the consistency of Eq. (18). Precisely the same reasoning applies if we approximate Δ_V and Δ_A by sums over meson poles, and approximate the left-hand side of Eq. (18) by a double sum over these poles.

Our new sum rules (1) and (4) are distinguished from those of the Adler-Weisberger type, in that they do not seem to have anything to do with low-energy theorems, but deal instead with high-energy behavior and, surprisingly, with the Schwinger terms. Another distinguishing feature of practical importance is that the integrals in (1) and (4) receive contributions only from states of fixed spin and isospin; this is presumably why our assumption of ρ and $A1$ saturation works so well.

The methods used here can obviously be applied to the currents of larger groups, like $SU(3) \otimes SU(3)$, and to the higher n-point functions of the currents.

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[1]M. Gell-Mann, Physics 1, 63 (1964).

[2]J. Schwinger, Phys. Rev. Letters 3, 296 (1959).

[3]That is, we shall treat chirality here as a symmetry of the Lagrangian which, though exact, does not leave the vacuum invariant; see Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961), etc. We could instead keep $m_\pi \neq 0$ and use partial conservation of axial-vector current, but since for once we not are directly concerned with soft pions it is simplest to take $m_\pi = 0$ from the beginning. The corrections are expected to be of the order m_π^2/m^2. Even if the currents were not conserved or partially conserved, the V and A Schwinger terms would still be equal; H. Schnitzer and S. Weinberg, to be published.

[4]These are the most general forms of the Källén-Lehmann representation for conserved currents, except that in (2) we use the fact that there is no $I=1$ scalar meson of zero mass, while we have to keep the one-pion contribution in (3).

[5]In our notation $V_a^\mu(x)$ and $A_a^\mu(x)$ are normalized to have commutation relations like those of $-i\bar{\psi}\gamma^\mu\tau_a\psi$ and $-i\bar{\psi}\gamma_5\gamma^\mu\tau_a\psi$, so our currents are twice those used by most other authors. The Goldberger-Treiman relation is thus $F_\pi = (2m_N/G)(g_A/g_V)$.

[6]This assumption will hold approximately to the extent that the currents are dominated by a finite number of meson poles in the sense of M. Gell-Mann and F. Zachariasen, Phys. Rev. 124, 953 (1961). Their work also suggests that our assumption about the high-momentum limit, and hence Eq. (4), may become exact, if $J=1$ fields are coupled to the currents [as advocated by J. J. Sakurai, Ann. Phys. (N.Y.) 11, 1 (1960)] and we let their bare mass go to infinity. If the ρ and $A1$ dominate ρ_V and ρ_A then Eq. (4) says that the coupling of the ρ and $A1$ to the V and A currents are the same, a result derived independently by D. Geffen (to be published).

[7]This formula was originally obtained by the author in the course of developing a Lagrangian, involving 1^+ and 1^- Yang-Mills fields, which reproduces the soft-pion results of current algebra. Note that if there were no massless pion, or if it decoupled from other particles so that $F_\pi = 0$, then we would recover the prediction of naive chirality, i.e., $m_A = m_\rho$. This is one more instance of the Goldstone theorem.

[8]K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters 16, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. 147, 1071 (1966); J. J. Sakurai, Phys. Rev. Letters 17, 552 (1966); M. Ademollo, to be published.

[9]A. H. Rosenfeld et al., University of California Radiation Laboratory Report No. UCRL-8030 (unpublished). The fact that a $\rho\pi$ enhancement is observed at $s = 2m_\rho^2$ does not necessarily force us to the conclusion that this is a single pure resonant state; all that is needed to derive Eq. (9) is that there is some particularly strong interaction in 1^+ $I=1$ states near this energy, which dominates the axial-vector spectral function. In this form the assumption of $A1$ dominance is supported by data from neutrino experiments, since the best fit to the axial-vector form factor uses an effective 1^+ meson mass significantly larger than m_ρ: S. Adler, private communication. There is some reason to be cautious in calling the $A1$ a pure resonance, because using current algebra to calculate the π-V-A vertex at zero pion energy, and using ρ and $A1$ dominance of the currents, yields a π-ρ-$A1$ coupling constant so large that the $A1$ width would be about 800 MeV, instead of the observed 125 MeV: D. Geffen, to be published; B. Renner, Phys. Letters 21, 453 (1966); H. Schnitzer, unpublished. [This result can also be immediately obtained from Eq. (18).] One way out of this difficulty is to suppose that the $A1$ is not a simple resonance, and that a Breit-Wigner representation of the $A1$ requires many levels. A more attractive possibility is that the π-ρ-$A1$ coupling constant that actually governs the rate of $A1$ decay, where the π is not soft, is much less than that calculated for zero-energy pions. The problem of the $A1$ width is under further study, in collaboration with H. Schnitzer. We would like to stress the importance of an experimental clarification of the nature of the $A1$.

[10]A formula for the Schwinger terms as integrals over spectral functions was derived by K. Johnson, Nucl. Phys. 25, 431 (1961); see also S. Okubo, Nuovo Cimento 44B, 1015 (1966). The special form of Eq. (17) arises because the current is conserved, but we keep a 0^- state at zero mass.