es a significant correction (of opposite sign). The relevant numerical results have been displayed in Table I.

The best reconciliation of the Desai model[13] with current data[1] on the K_1^0-K_2^0 mass difference would appear to indicate values of λ between -0.1 and -0.15 in the range of acceptable values[14] of $f_\rho^2/4\pi$, and a large S-wave, $I=0$, scattering length as well. However, one is likely straining the reliability of the model to draw so quantitative an inference. At the same time, one sees that this more realistic model does not require a large π-π phase shift[15] at energies in the neighborhood of the K-meson mass as has been suggested.[2]

We wish to thank Professor D. Harrington for his interest.

---

*Research supported in part by the National Science Foundation.

[1]Proceedings of the Thirteenth International Conference on High Energy Physics, Berkeley, California, 1966 (to be published).

[2]T. N. Truong, Phys. Rev. Letters 17, 1102 (1966).

[3]V. Barger and E. Kazes, Phys. Rev. 124, 279 (1961).

[4]K. Nishijima, Phys. Rev. Letters 12, 39 (1964).

[5]Briefly, these are that (a) the effect of CP noninvariance, (b) the contributions of the leptonic and three-pion ($K_2^0 \to 3\pi$) modes, and (c) the pion and η^0 pole contributions to the self-energy of K_2^0 can be neglected.

[6]We work in units $m_\pi = 1$.

[7]We set $\nu_K = \frac{1}{4}(m_K^2/m_\pi^2) - 1$; as usual, we take

$$\left(\frac{\nu}{\nu+1}\right)^{1/2} \cot\delta_0^{\ I}(\nu) = \mathrm{Re}[A_0^{\ I}(\nu)]^{-1} = \frac{\mathrm{Re}D_0^{\ I}(\nu)}{N_0^{\ I}(\nu)},$$

where

$$D_0^{\ I}(\nu) = 1 - \frac{\nu-\nu_0}{\pi}\int_0^\infty d\nu' \left(\frac{\nu}{\nu'+1}\right)^{1/2} \frac{N_0^{\ I}(\nu')}{(\nu'-\nu_0)(\nu'-\nu)},$$

with $\nu_0 = -\frac{2}{3}$.

[8]S. L. Adler, Phys. Rev. 137, B1022 (1965); 139, B1638 (1965).

[9]J. J. Sakurai, Phys. Rev. Letters 17, 552 (1966).

[10]However, in contrast to Weinberg's results [S. Weinberg, Phys. Rev. Letters 17, 616 (1966)], in this model we find $A = 0$, $B = -\frac{1}{2}C$.

[11]It is interesting that in Weinberg's model one finds for λ the minuscule value $\lambda = -0.0028(f_\rho^2/4\pi)$.

[12]$\beta_{01} = 1$, $\beta_{21} = -\frac{1}{2}$. See, for example, L. A. P. Balasz, Phys. Rev. 128, 1939 (1962).

[13]B. R. Desai, Phys. Rev. Letters 6, 497 (1961).

[14]J. J. Sakurai, Phys. Rev. Letters 17, 1021 (1966).

[15]Indeed for $f_\rho^2/4\pi = 2.16$, and $\lambda = 0$, the unitarization provided by Desai's model reduces the Born phase shift, $\delta_0^{\ 0}(\nu_K)$, very slightly.

S-WAVE $\pi\pi$ SCATTERING AND THE K_1^0-K_2^0 MASS DIFFERENCE*

Kyungsik Kang
Department of Physics, Brown University, Providence, Rhode Island

and

David J. Land†
Radiation Physics Division, U. S. Naval Ordnance Laboratory, Silver Spring, Maryland

(Received 15 February 1967)

The result of recent measurements of the K_1^0-K_2^0 mass difference has been used as a test of various solutions to s-wave $\pi\pi$ scattering which are obtained by solving the full N/D equations with various given forms of the driving force. Certain features of these solutions with regard to the evaluation of the mass difference are also pointed out.

In view of the fact that several recent measurements of the K_1^0-K_2^0 mass difference have resolved some of the previous experimental uncertainties,[1] it is of some interest to consider this problem again theoretically.[2-5] These measurements indicate a value of $\Delta M \equiv M(K_1^0) - M(K_2^0) \simeq -0.5\tau_1^{-1}$, where τ_1 is the lifetime of the K_1^0 meson. In the present work, we use the K_1^0-K_2^0 mass difference as a test of various solutions to the problem of $I=0$, s-wave $\pi\pi$ scattering. We obtain some exact solutions for the s-wave amplitude by solving the full N/D equations with various given forms of the driving force. We then discuss certain features of these solutions with regard to the evaluation ΔM.

Since the K_1^0-K_2^0 mass difference is produced by the weak interactions, the problem reduces to a calculation of the self-energies of the K_1^0 and K_2^0 mesons due to the weak interactions. From experiments, the lifetime of K_1^0 is of the order of 10^{-10} sec while that of K_2^0 is of the order of 10^{-7} sec. In addition, the leptonic decay rates for the K_1^0 and K_2^0 are known to be negligible as compared with the two-pion decay mode of the K_1^0 which is predominantly in the $I=0$ state, according to the $|\Delta I| = \frac{1}{2}$ rule. Thus the mass difference may be due primarily to the self-energy of the K_1^0, arising from the two-pion state with $I=0$.

The self-energy operator for the K_1^0 meson, $\Sigma(s)$, is assumed to obey an unsubtracted dispersion relation[2,3]:

$$\Sigma(s) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} ds' \frac{\mathrm{Im}\Sigma(s')}{s'-s}. \tag{1}$$

From unitarity, $\mathrm{Im}\Sigma(s)$ is related to the $K_1^0\pi\pi$ form factor $F(s)$ by the equation

$$\mathrm{Im}\Sigma(s) = c\left(\frac{s-4\mu^2}{s}\right)^{1/2} |F(s)|^2, \tag{2}$$

where c is a constant. The form factor itself is related to the $I=0$, s-wave $\pi\pi$ scattering amplitude by

$$F(s) = [D(s)]^{-1}, \tag{3}$$

where $D(s)$ is the denominator function in the N/D decomposition of the s-wave $\pi\pi$ amplitude. The mass difference, ΔM, is given by

$$2\tau_1 \Delta M = -\mathrm{Re}\Sigma(M^2)\,\mathrm{Im}\Sigma(M^2), \tag{4}$$

where M is the mass of either the K_1^0 or the K_2^0 meson.

In the present note, we determine ΔM from the above equations by using some exact solutions to the problem of $\pi\pi$ scattering. These solutions are obtained from the once-subtracted full N/D equations, in which we take as the driving forces the exchange of the ρ meson and the exchange of either an $I=0$, s-wave resonance or an $I=0$, s-wave nonresonant amplitude, characterized by a particular value of the scattering length and represented by the effective-range formula.[6] We demand that these solutions be approximately self-consistent, in the sense that an output resonance have approximately the same parameters as the assumed input resonance or that a nonresonant solution be characterized by the same scattering length as the nonresonant exchange amplitude. Because of the exchange of the vector meson, a cutoff in the N/D equations is necessary.

It should be noted that the integral for $\Sigma(s)$ in Eq. (1) is convergent, when the N/D solutions to s-wave $\pi\pi$ scattering are used in its evaluation. However, several cutoffs are used in the integral in order to determine the relative contributions to ΔM from the various regions of integration. We find that about 90% of ΔM comes from the low-energy region of the integration below 1 GeV, as is illustrated in Tables I and II. This aspect is quite satisfactory in a low-energy calculation which puts emphasis on the nearby singularities of the S matrix.

Table I. Values of the K_1^0-K_2^0 mass difference, for several values of the self-consistent scattering length and for an exchanged ρ meson with a width of 90 MeV and a width of 128 MeV. R_0 is chosen to be 0. The corresponding values of the conventional pion-pion coupling constant, λ, $-\frac{1}{2}\cot\delta(M^2)$, and $\delta(M^2)$ are also presented. A cutoff of $66\mu^2$ is used in the N/D equation. Two values of the cutoff Λ are used in the evaluation of $\Sigma(s)$.

Γ_ρ (MeV)	a_0	-5λ	$\Delta M\tau_1$ $\Lambda=66\mu^2$	$\Delta M\tau_1$ $\Lambda=1000\mu^2$	$-\frac{1}{2}\cot\delta(M^2)$	$\delta(M^2)$ (deg)
90	1.0	0.400	0.169	0.154	−0.356	55
90	0.5	0.163	0.118	0.103	−0.438	49
90	0.2	−0.024	0.057	0.043	−0.562	42
90	−0.2	−0.350	−0.185	−0.203	−1.889	15
90	−0.3	−0.457	−0.317	−0.341	−9.953	3
90	−0.4	−0.583	−0.514	−0.554	2.720	−10
90	−0.5	−0.733	−0.843	−0.942	1.131	−24
128	1.0	0.170	0.445	0.437	−0.122	76
128	0.5	−0.034	0.414	0.407	−0.144	74
128	0.2	−0.195	0.372	0.366	−0.175	71
128	−0.2	−0.459	0.146	0.142	−0.415	50
128	−0.3	−0.547	0.020	0.016	−0.682	36
128	−0.4	−0.653	−0.164	−0.169	−1.757	16
128	−0.5	−0.782	−0.460	−0.470	3.369	−9
128	−0.6	−0.943	−1.036	−1.072	0.871	−30

Table II. Values of the K_1^0-K_2^0 mass difference for several s-wave, $I=0$ resonances of mass M_R but with the same input width 90 MeV. For $M_R \geq 550$ MeV, the s-wave amplitude is self-consistent in mass only. The ρ parameters used are a width of 90 MeV and a mass of 769 MeV. ΔM does not depend on the values of the cutoff Λ used to evaluate $\Sigma(s)$ for $\Lambda \geq 264\mu^2$. A cutoff of $264\mu^2$ is used in the N/D equations as well as in $\Sigma(s)$.

M_R (MeV)	a_0	-5λ	$\Delta M\tau_1$	$-\frac{1}{2}\cot\delta(M^2)$	$\delta(M^2)$ (deg)
395	−0.159	−0.500	1.25	0.222	−66
550	−0.540	−0.855	−0.02	−0.500	45
575	−0.562	−0.886	−0.30	−2.034	14
592	−0.579	−0.905	−0.54	11.773	−2
655	−0.605	−0.955	−1.35	1.060	−25

In Table I, we present the results of our calculations for the case of the nonresonant s-wave amplitude. We see that, in order to obtain the value $\Delta M \simeq -0.5\tau_1^{-1}$, we require a negative value of the $I=0$, s-wave scattering length; the phase shift for this solution is negative in the low-energy region and becomes positive at higher energies, with a value of $0° \pm 10°$ around the energy of the K-meson mass (497 MeV). We also observe from these solutions that a positive value of scattering length leads to a positive value for ΔM and a phase shift at the K mass of around 30 to 70°. A similar feature was also observed in the work of Barger and Kazes[2] in which they assumed various forms for the $\pi\pi$ phase shift.

In the case of a resonant s-wave amplitude, we obtain a self-consistent resonance with a mass less than or equal to 400 MeV.[7] Although it is possible to get the resonance with a self-consistent mass only up to about 700 MeV, the output width is larger than the input width for values of the resonant mass greater than 550 MeV, by a factor as large as, say, 5 for M_R = 655 MeV. We present the results in Table II including those of the resonant s-wave solutions with a self-consistent mass only. We note, as has been observed previously,[3] that mass values of the $\pi\pi$ resonance below (above) the K mass produce positive (negative) values for ΔM. We notice that the phase shift at the K mass depends sensitively on the output mass of the s-wave resonance.

We have also performed the calculations by including the exchange of an $I=2$, s-wave, nonresonant amplitude, again represented by the effective-range formula with $a_2 = -0.06$ and $R_2 = 0$. The results are essentially unchanged by the presence of the $I=2$, s-wave exchange. In addition, the results are also insensitive to R_I with $|R_I| \leq 0.5$.[8]

There are certain features of the solutions which we wish to stress. The dominant contribution to the Born term arises from the exchange of the ρ meson and, furthermore, this force is extremely strong, as evidenced by the fact that many (unacceptable) solutions to the s-wave amplitude exhibit either a bound state or a ghost state.[9] This point has been observed by other authors[10] too. On this account the behavior of the N function follows the general character of the ρ-exchange Born term. Both of these functions in the low-energy region are monotonic increasing with negative second derivative and both have a zero in the vicinity of the physical threshold; the zero of the ρ-exchange Born term occurs at $s = -m_\rho^2/2 + 2\mu^2$, while the zero of the full amplitude occurs closer to or actually in the physical region. We illustrate these points for a typical case in the graph of Fig. 1, in which we present the total Born term, the Born term for ρ exchange alone, and the N function. In all cases, we find from our solutions that the N function has a detailed structure. This is in contrast to the situation in which the N function is set equal to a constant and the D function is chosen to contain the desired character of the amplitude.[5]

It has been pointed out that, by considering the analytic properties of the function $[N(s)D(s)]^{-1}$, one obtains the equation[5]

$$\frac{1}{N(s)D(s)} = \Sigma(s) + \frac{1}{\pi}\int_L ds' \, \frac{[1/D(s')]\,\mathrm{Im}[1/N(s')]}{s'-s}, \quad (5)$$

where L represents the left-hand cut of the amplitude. From Eq. (5) one immediately has for the K_1^0-K_1^0 mass difference

$$\Delta M\tau_1 = -\tfrac{1}{2}\cot\delta(M^2) + \text{correction term due to left-hand integral}, \quad (6)$$

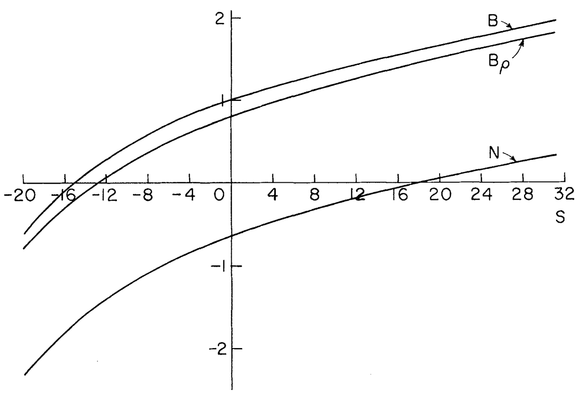

FIG. 1. Plot of the total Born term $B(s)$, the Born term for ρ exchange alone $B_\rho(s)$, and N function $N(s)$, for an amplitude with ρ exchange ($\Gamma_\rho = 90$ MeV) and the exchange of a nonresonant $I=0$, s-wave amplitude, characterized by the scattering length $a_0 = -0.4$. Portions of $B(s)$ and $N(s)$ below $s=0$ represent their real parts only. The unit $\mu = 1$ is used.

where $\delta(s)$ is the phase shift of the $I=0$, s-wave, $\pi\pi$ scattering amplitude. Equation (5) is valid as it stands only if N and D have no zeros; zeros of N and D contribute as poles to Eq. (5) and must therefore be included in the correction term in the evaluation of ΔM of Eq. (6).

We remark that our evaluation of ΔM, either by calculating the self-energy $\Sigma(s)$ directly from the integral of Eq. (1), or by calculating $\Sigma(s)$ from Eq. (5) with all correction terms included, gives the same numerical results. In Tables I and II, we include the value of $-\frac{1}{2}\cot\delta(M^2)$ in order to illustrate the effect of these correction terms. It is interesting to notice from Tables I and II that $-\frac{1}{2}\cot\delta(M^2)$ alone can give a misleadingly attractive result for the mass difference. In particular, Table I shows that for solutions with a positive scattering length $-\frac{1}{2}\cot\delta(M^2)$ has not only the correct sign but also a reasonable magnitude of the mass difference.

If the N function is indeed constant, Eq. (6) is correct with no correction term. However, our solutions show that the N function is not constant but rather has a characteristic behavior, and in particular, it has a zero near or in the physical region. Thus, we feel that a calculation in which the N function is taken simply as constant, or, more generally, in which the correction terms in Eq. (6), both for the integral along the left-hand cut and for possible zeros in N or D, are ignored, is apt to yield misleading results.

We would like to thank Dr. T. N. Truong for many stimulating conversations. In addition, one of us (D. J. L.) wishes to thank the Computing Laboratory at the U. S. Naval Ordnance Laboratory for their assistance.

---

*Work supported in part by the U. S. Atomic Energy Commission, Report No. NYO-2262TA-144.

†National Academy of Sciences–National Research Council Resident Research Postdoctoral Associate.

[1]A summary of measurements of the K_1^0-K_2^0 mass difference can be found in Proceedings of the Thirteenth International Conference on High Energy Physics, Berkeley, California, 1966 (to be published).

[2]V. Barger and E. Kazes, Phys. Rev. 124, 279 (1961).

[3]K. Nishijima, Phys. Rev. Letters 12, 39 (1964).

[4]S. Patil, Phys. Rev. Letters 13, 454 (1964).

[5]T. N. Truong, Phys. Rev. Letters 17, 1102 (1966), in which further references can be found.

[6]The effective range formula for the s waves is given by $q\cot\delta_I = 1/a_I + \frac{1}{2}R_I q^2$; $I=0$ or 2, and q is the barycentric momentum related to s by $s = 4(q^2+\mu^2)$, μ being the mass of the pion. We use the units $\hbar = c = 1$ throughout.

[7]It is interesting to note that the recent analysis of the πp backward scattering by G. Lovelace, R. M. Heinz, and A. Donnachie, Phys. Letters 22, 332 (1966), suggests also an s-wave di-pion resonance around 430 MeV.

[8]For the case of $a_0 = 0.2$, $R_0 = 0$, $a_2 = -0.06$, $R_2 = 0$, we find $-5\lambda = -0.020$, $\Delta M\tau_1 = 0.050$ for $\Lambda = 66\mu^2$, $\Delta M\tau_1 = 0.036$ for $\Lambda = 1000\mu^2$, $-\frac{1}{2}\cot\delta(M^2) = -0.576$, and $\delta(M^2) = 42°$ while for the case of $a_0 = 0.2$, $R_0 = 0.4$, $a_2 = -0.06$, $R_2 = -0.4$, the corresponding results are $-5\lambda = -0.023$, $\Delta M\tau_1 = 0.055$ for $\Lambda = 66\mu^2$, $\Delta M\tau_1 = 0.041$ for $\Lambda = 1000\mu^2$, $-\frac{1}{2}\cot\delta(M^2) = -0.564$, and $\delta(M^2) = 42°$.

[9]For this reason we perform the calculations by using a ρ meson with a width of 90 MeV and the currently accepted value of 128 MeV. The mass of the ρ is taken to be 769 MeV.

[10]P. D. B. Collins, Phys. Rev. 142, 1163 (1966); C. F. Kyle, A. W. Martin, and H. R. Pagels, to be published.