[6]See, for example, H. Muirhead, Physics of Elementary Particles (Pergamon Press, New York, 1965), p. 278.

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SUM RULES BASED ON CAUSALITY*

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It will be shown in this note that if an amplitude $A(x)$ is causal and its Fourier transform $\tilde{A}(q)$ satisfies the Dyson representation, then we can derive a physically meaningful sum rule provided it converges. As an example, we take the commutator of the isovector current densities $j_\mu{}^a(x)$ $(a=1,2,3)$,

$$\tilde{A}_{\mu\nu}{}^{ab}(q)=\int d^4x\, e^{iq\cdot x}\langle p|[j_\mu{}^a(x),j_\nu{}^b(0)]|p\rangle, \quad (1)$$

where the state $|p\rangle$ represents, for example, a proton of momentum p and the spin average is assumed. Because of the conservation law, we have two independent amplitudes, and we write

$$\tilde{A}_{\mu\nu}{}^{ab}(q)=\sum_{i=1,2}L_{\mu\nu}{}^{(i)}(q)\tilde{A}_i{}^{ab}(q)$$

with

$$L_{\mu\nu}{}^{(1)}(q)=m^{-2}[q^2p_\mu p_\nu-(p_\mu q_\nu+p_\nu q_\mu)p\cdot q+g_{\mu\nu}(p\cdot q)^2]$$

and

$$L_{\mu\nu}{}^{(2)}(q)=q^2q_{\mu\nu}-q_\mu q_\nu,$$

where m is the proton mass. We further decompose $\tilde{A}_i{}^{ab}$ into isosymmetric and antisymmetric parts by

$$\tilde{A}^{ab}(q)=\delta_{ab}\tilde{A}^{(s)}(q)+i\epsilon_{abc}\tau_c\tilde{A}^{(a)}(q). \quad (2)$$

The amplitude $q^2A_1{}^{(a)}$ satisfies the ordinary Dashen–Gell-Mann[1] and Fubini[2]-type sum rule. The other amplitudes $A_i{}^{(s)}(x)$ $(i=1,2)$ and $\tilde{A}_2(x)$ can be shown[3] to be causal, vanishing outside the light cone, and their Fourier transforms $\tilde{A}_i{}^{(s)}(q)$ and $\tilde{A}_2{}^{(a)}(q)$ satisfy the Dyson representation,[4] which is

$$\tilde{A}(q)=\int_0^\infty ds\int d^4u\,\epsilon(q-u)\delta[(q-u)^2-s]\psi(u,s)]. \quad (3)$$

The immediate consequence of this is[5]

$$\int_{-\infty}^{\infty}dq_0\,A(q)=0. \quad (4)$$

Now, Eq. (4) gives us the sum rule in the following way: The isosymmetric amplitudes $\tilde{A}_i{}^{(s)}$ have the crossing symmetry $\tilde{A}_i{}^{(s)}(q)=-\tilde{A}_i{}^{(s)}(-q)$. Choosing as invariant variables $\lambda=q^2$ and $\nu=p\cdot q$, we write $\tilde{A}_i{}^{(s)}(q)=\tilde{A}_i{}^{(s)}(\lambda,\nu)$. Changing the integration variable from q_0 to ν, Eq. (4) gives in this case

$$\int_0^\infty d\nu[\tilde{A}_i{}^{(s)}(\lambda_+,\nu)-\tilde{A}_i{}^{(s)}(\lambda_-,\nu)]=0,$$

where

$$\lambda_\pm=[\nu^2\pm 2\vec{p}\cdot\vec{q}\,\nu+(\vec{p}\cdot\vec{q})^2]p_0{}^{-2}-\vec{q}^2.$$

In order that the above equation holds in any reference frame, the first-order term in $\vec{p}\cdot\vec{q}$ must vanish. Taking the limit $|\vec{p}|\to\infty$ in the Taylor expansion, we obtain

$$\int_0^\infty d\nu\,\nu\frac{\partial}{\partial\lambda}\tilde{A}_i{}^{(s)}(\lambda,\nu)=0 \quad (i=1,2), \quad (5)$$

where λ is an arbitrary negative number independent of ν. For the antisymmetric part $\tilde{A}_2{}^{(a)}$, we have the crossing symmetry $\tilde{A}_2{}^{(a)}(q)=\tilde{A}_2{}^{(a)}(-q)$, and Eq. (4) for $\tilde{A}_2{}^{(a)}$ gives, again in the limit of $|\vec{p}|\to\infty$,

$$\int_0^\infty d\nu\,\tilde{A}_2{}^{(a)}(q)=0, \quad (6)$$

which has the form of what is called the superconvergence sum rule.[6]

Before discussing the sum rules (5) and (6), we remark that we could have taken the electromagnetic current j_μ instead of the isovector current in Eq. (1) and that the amplitudes

$\tilde{A}_i^Q(q)$ thus defined have the same properties as $\tilde{A}_i^{(s)}(q)$, especially satisfying Eq. (5), which we could write in the form

$$\int_0^\infty d\nu\,\nu[\tilde{A}_i^Q(\lambda_1,\nu)-\tilde{A}_i^Q(\lambda_2,\nu)]=0, \tag{7}$$

where λ_1 and λ_2 are two arbitrary negative numbers. $\tilde{A}_i^Q(\lambda,\nu)$ are related to the total cross sections of the longitudinal and transverse photons of mass $\sqrt{\lambda}$ on a proton, say, σ_L and σ_T by

$$\tilde{A}_1^Q(\lambda,\nu)=\frac{2m}{e^2(\nu^2-m^2\lambda)^{1/2}}[\sigma_L(\lambda,\nu)-\sigma_T(\lambda,\nu)]$$

and

$$\tilde{A}_1^Q(\lambda,\nu)+\tilde{A}_2^Q(\lambda,\nu)=-\frac{2(\nu^2-m^2\lambda)^{1/2}}{e^2m\lambda}\sigma_L(\lambda,\nu). \tag{8}$$

Separating the pole contribution, we can write (7) for $i=1$ in the form

$$F_{1p}^2(\lambda_1)-\lambda_1F_{2p}^2(\lambda_1)-(\lambda_1\to\lambda_2)+\frac{1}{2\pi^2\alpha}\int_{\nu_0}^\infty d\nu\left\{\frac{\nu}{(\nu^2-m^2\lambda_1)^{1/2}}[\sigma_T(\lambda_1,\nu)-\sigma_L(\lambda_1,\nu)]-(\lambda_1\to\lambda_2)\right\}=0, \tag{9}$$

where F_{1p} and F_{2p} are the charge and moment form factors of the proton. The threshold ν_0 is determined by $2\nu_0=2m\mu-\mu^2-\mathrm{Sup}(\lambda_1,\lambda_2)$, where μ is the pion mass. It should be noted that the integrand involves the difference of two cross sections for two different masses so that the integral may converge even if the cross sections go to constants at infinity. The sum rule (9) could be checked experimentally since $\sigma_{T,L}(\lambda,\nu)$ for $\lambda<0$ can be obtained from the cross section for the inelastic electron scattering by a proton, in which only the scattered electron is detected,[7] i.e., in the laboratory system:

$$d\sigma=\frac{\alpha}{2\pi^2}dk'\,d\Omega\frac{k'}{k}\frac{|\vec{q}|}{(-\lambda)}\frac{1}{1-\epsilon(\theta)}[\sigma_T(\lambda,\nu)-\epsilon(\theta)\sigma_L(\lambda,\nu)], \tag{10}$$

where k and k' are the momenta of the incident and scattered electron, $q=k-k'$ being the momentum of the virtual photon and

$$\epsilon(\theta)=\frac{(-\lambda/\vec{q}^2)\cot^2(\tfrac{1}{2}\theta)}{2-(\lambda/\vec{q}^2)\cot^2(\tfrac{1}{2}\theta)}.$$

The Eq. (7) for $A_1^Q+A_2^Q$ is simpler than that for A_2^Q alone, and using (8) we have

$$\left[F_{1p}(\lambda_1)+\frac{\lambda_1}{2m}F_{2p}(\lambda_1)\right]^2-[\lambda_1\to\lambda_2]^2+\frac{1}{2\pi^2\alpha}\int_{\nu_0}^\infty d\nu\left[\frac{\nu(\nu^2-m^2\lambda_1)^{1/2}}{m^2\lambda_1}\sigma_L(\lambda_1,\nu)-(\lambda_1\to\lambda_2)\right]=0. \tag{11}$$

The convergence of the integral will be worse than that of (9).

Going back to the sum rules for the isovector amplitudes (5) and (6), we can express the amplitudes $\tilde{A}_i^{(s),(a)}$ in terms of the cross sections of the isovector photons with ± charges, $\sigma^\pm$, as in (8). The latter can be converted to the cross sections related to the isovector part of a photon. Namely, by the rotation in isospin space, we have $\sigma^--\sigma^+=2\sigma^{(a)}=2\sigma_{1/2}^0-\sigma_{3/2}^0$ and $\sigma^-+\sigma^+=2\sigma^{(s)}=2\sigma^0$, where σ^0 is the total cross section of the neutral isovector photon on a proton and $\sigma_{1/2}^0$ and $\sigma_{3/2}^0$ are $I=\frac{1}{2}$ and $\frac{3}{2}$ part of the same cross section. The sum rule (5) for $i=1$ gives for $\lambda=0$

$$F_{1v}'(0)-\frac{1}{2}F_{2v}^2(0)+\frac{1}{2\pi^2\alpha}\int_{\nu_0}^\infty d\nu\frac{m^2}{\nu^2}\sigma_T^0(0,\nu)+\frac{1}{\pi^2\alpha}\int_{\nu_0}^\infty d\nu\left\{\frac{\partial}{\partial\lambda}[\sigma_T^0(\lambda,\nu)-\sigma_L^0(\lambda,\nu)]\right\}_{\lambda=0}=0, \tag{12}$$

where F_{1v} and F_{2v} are the isovector part of the charge and moment form factors of the nucleon, normalized as $F_{1v}(0)=1$ and $F_{2v}(0)=(\mu_p-\mu_n)/2m$. For comparison, the Cabibbo-Radicati sum rule[8] is given in our notation by

$$\left[\frac{\partial}{\partial\lambda}\int_{\lambda_0}^\infty d\nu\,\lambda\tilde{A}_1^{(a)}(\lambda,\nu)\right]_{\lambda=0}=0 \tag{13}$$

or

$$F_{1v}'(0)-\frac{1}{2}F_{2v}^2(0)-\frac{1}{4\pi^2\alpha}\int_{\nu_0}^\infty\frac{d\nu}{\nu}(2\sigma_{1/2}^{0},{}_T-\sigma_{3/2}^{0},{}_T)=0. \tag{14}$$

At the moment we do not have enough experimental information on σ_T, $L^0(\lambda, \nu)$ to test the sum rule (12). In the pole approximation taking into account only the nucleon and $N^*(1238)$, Eq. (12) is replaced by

$$F_{1v}'(0)-\frac{1}{2}F_{2v}^2(0)-\frac{4\beta^2(0)}{3\mu^2}+\frac{2\beta(0)\beta'(0)}{3\mu^2m^{*2}}(m^{*2}-m^2)(m^2+3m^{*2})=0, \quad (15)$$

where m^* and μ are the masses of N^* and the pion, respectively. $\beta'=d\beta(\lambda)/d\lambda$ and $\beta(\lambda)$ is the coupling constant defining the magnetic dipole transition from a nucleon to N^*, and according to Gourdin and Salin[9] and also to Mathews,[10] has the value[11] $\beta(0)=0.3$. The value of β'/β, determined from (18), is insensitive to the change of the value of β because the first two terms of (15) almost cancel each other. Thus we obtain as the rms radius of the N-N^* transition current $(6\beta'/\beta)^{1/2}\sim 0.4\times10^{-13}$ cm as long as $\beta(0)\gtrsim 0.2$. This is nice because we know that the N^* contribution to the Cabibbo-Radicati sum rule (14) in the pole approximation using the value $\beta=0.3$ is twice as big as the value obtained by actually integrating the $I=\frac{3}{2}$ cross section up to 500 MeV (Gilman and Schnitzer[12]). Our sum rule is free of this bad feature of the narrow resonance approximation.

The sum rule for $A_1^{(s)}+A_2^{(s)}$ is obtained from (5) and (8) and gives

$$F_{1v}'(0)+\frac{1}{2m}F_{2v}(0)-\frac{1}{2\pi^2\alpha}\int_{\nu_0}^\infty d\nu\left\{\left(1-\frac{2\nu}{m^2}\frac{\partial}{\partial\lambda}\right)\frac{\sigma_L^{\ 0}}{\lambda}\right\}_{\lambda=0}=0. \quad (16)$$

In the pole approximation we have

$$F_{1v}'(0)+\frac{1}{2m}F_{2v}(0)-\frac{2\beta^2}{3\mu^2m^2}m^*(m^*-m)+\frac{2\beta\beta'}{3\mu^2m^2}(m^{*2}-m^2)(m^*-m)^2=0. \quad (17)$$

In order to obtain a positive β' we have to choose $\beta\sim 0.44$, so that this sum rule is not reliable, at least in the pole approximation. The superconvergence sum rule (6) gives

$$mF_{2v}(0)+m^2F_{2v}^{\ 2}(0)-\frac{1}{2\pi^2\alpha}\int_{\nu_0}^\infty\frac{d\nu}{\nu}\left[\frac{\nu^2}{\lambda}(2\sigma_{1/2,L^0}-\sigma_{3/2,L^0})-m^2(2\sigma_{1/2,T^0}-\sigma_{3/2,T^0})\right]_{\lambda=0}=0, \quad (18)$$

which reduces in the pole approximation to

$$mF_{2v}(0)+m^2F_{2v}^{\ 2}(0)-\frac{\beta^2}{3\mu^2m^{*2}}[2m(m^3+m^{*3})-(m^{*2}-m^2)^2]=0. \quad (19)$$

Using the experimental value for $F_{2v}(0)$, we find $\beta^2=0.1$. Conversely, $\beta=0.3$ gives $\mu_p-\mu_n=35$. Thus, the sum rule (19) seems to work remarkably well, although this could be accidental.

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$$\lim_{\Lambda\to\infty}\int_{-\infty}^{\infty}dq_0\exp(-q_0^2/\Lambda^2)\tilde{A}(q)=0,$$

which is valid only when

$$\lim_{s\to\infty}\psi(u,s)=0,$$

as shown in Ref. (3).

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