FUBINI SUM RULE AND ANALYTICITY IN ANGULAR MOMENTUM PLANE

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Recently Fubini, Furlan, and Rossetti,[1] using the following equal-time commutation relations for vector currents,

$$[j_0^{\alpha}(x), j_0^{\beta}(y)]\delta(x^0-y^0)=\epsilon^{\alpha\beta\gamma}j_0^{\gamma}(x)\delta^4(x-y) \quad (1)$$

$(\alpha, \beta, \gamma=$ isotopic indices),

have derived the sum rule

$$\frac{1}{\pi}\int ds' a_1^{\alpha\beta}(s', t; K_1^2, K_2^2)=\epsilon^{\alpha\beta\gamma}F^{\gamma}(t). \quad (2)$$

The amplitude $a_1^{\alpha\beta}(s, t, K_1^2, K_2^2)$ is defined by the expansion, in terms of invariant amplitudes, of the quantity

$$t_{\mu\nu}^{\alpha\beta}=\int d^4x \exp[i(K_1+K_2)\cdot x] \times\langle p_2|[j_{\mu}^{\alpha}(\tfrac{1}{2}x), j_{\nu}^{\beta}(-\tfrac{1}{2}x)]|p_1\rangle \quad (3)$$

$(\mu, \nu=$ Minkowski indices),

given by

$$t_{\mu\nu}=a_1P_{\mu}P_{\nu}+a_2P_{\mu}K_{2\nu}+a_3P_{\mu}K_{1\nu} +a_4K_{2\mu}P_{\nu}+a_5K_{2\mu}K_{2\nu} +a_6K_{2\mu}K_{1\nu}+a_7K_{1\mu}P_{\nu} +a_8K_{1\mu}K_{2\nu}+a_9K_{1\mu}K_{1\nu}+a_{10}g_{\mu\nu}. \quad (4)$$

The quantity $t_{\mu\nu}^{\alpha\beta}$ is the absorptive part of the amplitude $T_{\mu\nu}^{\alpha\beta}$ which describes the process

$$\pi(p_1)+\gamma_{\alpha}(K_1)\to\pi(p_2)+\gamma_{\beta}(K_2),$$

where p_1, K_1, etc., are respective four-momenta and $\gamma_{\alpha}(K_1)$ stands for "charged photons" with isotopic index α, four-momentum K_1. We have $p_1+K_1=p_2+K_2$, $P=p_1-p_2$, $s=(p_1+K_1)^2$, $u=(p_1-K_2)^2$, $t=(K_1-K_2)^2$. The $F^{\gamma}(t)$ is the pion form factor

$$\langle p_2|j_{\mu}^{\gamma}(0)|p_1\rangle=F^{\gamma}(t)P_{\mu}. \quad (5)$$

The purpose of the present note is to study the analyticity in angular momentum plane of the definite helicity amplitudes for the process $\pi+\pi\to\gamma_{\alpha}+\gamma_{\beta}$. We shall restrict ourselves to $I=1$ t-channel (i.e., $\pi+\pi\to\gamma_{\alpha}+\gamma_{\beta}$) amplitude only. We shall further put $K_1^2=K_2^2=0$ for simplicity. We shall see that (i) it is inconsistent with Fubini sum rule to have only moving poles, i.e., Regge poles in these amplitudes, and that (ii) if one has a fixed $J=1$ pole, which does not contribute to the s-aborptive part of the amplitude, then one has the Fubini sum rule for $\mathrm{Re}\,\alpha_{\rho}(t)<1$, i.e., $t<m_{\rho}^2$ and real, and the inconsistency is avoided.

Let us expand the amplitude $T_{\mu\nu}$ in terms of invariant amplitudes. We have, just as for $t_{\mu\nu}$ given by the expression (4), in general,

$$T_{\mu\nu}=A_1P_{\mu}P_{\nu}+A_2P_{\mu}K_{2\nu}+\cdots+A_{10}g_{\mu\nu}. \quad (6)$$

If the current is conserved, i.e., $\partial^{\mu}j_{\mu}^{\alpha}=0$, we must have[2]

$$T_{\mu\nu}^{\alpha\beta}K_1^{\nu}=K_2^{\nu}T_{\nu\mu}^{\alpha\beta} =e^{\alpha\beta\gamma}F^{\gamma}P_{\mu}\equiv F^{\alpha\beta}(t)P_{\mu}. \quad (7)$$

Using current conservation conditions (7) and $K_1^2=K_2^2=0$, we can express all ten amplitudes in terms of two invariant amplitudes, say A and B, and $F(t)$. We finally obtain

$$T_{\mu\nu}^{(1)}=\left[P_{\mu}P_{\mu}+K_{1\mu}K_{2\nu}+\frac{s-u}{t}(K_{1\mu}P_{\nu}+P_{\mu}K_{2\nu})+\frac{2(s-m^2)(u-m^2)}{t}g_{\mu\nu}\right]A^{(1)}(s,t) +\left[g_{\mu\nu}-\frac{K_{1\mu}K_{2\nu}}{(K_1\cdot K_2)}\right]B^{(1)}(s,t)+[(s-u)g_{\mu\nu}-2(K_{1\mu}P_{\nu}+P_{\mu}K_{2\nu})](1/t)F^{(1)}(t), \quad (8)$$

where m is the pion mass and superscript 1 refers to isospin 1 in the t channel.

We next project out the definite helicity amplitudes $T_{\lambda_1\lambda_2}$ in the center-of-mass frame in the t

channel:

$$T_{\lambda_1\lambda_2} = T_{\mu\nu} \exp[(\lambda_2)\mu](K_2) \exp[(\lambda_1)\nu](K_1). \quad (9)$$

We obtain

$$T_{++} = 2m^2A(s,t) + B(s,t) + [(s-u)/t]F(t),$$
$$T_{+-} = 2A(s,t)p^2\sin^2\varphi, \quad (10)$$

where $s = -p^2 - q^2 + 2pq\cos\varphi$, $u = -p^2 - q^2 - 2pq\cos\varphi$, $t = 4q^2 = 4(p^2 + m^2)$. Using the Jacob-Wick formalism, we have the expansion

$$T_{++} = (1/q)\sum_J (J+\tfrac{1}{2})T_{++}^J(t)P_J(\cos\varphi),$$

$$T_{+-} = \frac{1}{q}\sum_{J>1}(J+\tfrac{1}{2})T_{+-}^J(t)d_{20}^J(\varphi). \quad (11)$$

Hence we have the partial wave expansion

$$A(s,t) = \frac{1}{2p^2q}\sum_{q>1}(J+\tfrac{1}{2})T_{+-}^J(t)$$
$$\times[(J+2)(J+1)J(J-1)]^{-1/2}P_J''(\cos\varphi), \quad (12)$$

$$B(s,t) + 2m^2A(s,t)$$
$$= (1/q)\sum_J (J+\tfrac{1}{2})[T_{++}^J(t) - \tfrac{2}{3}pF(t)\delta_{J1}]P_J(\cos\varphi). \quad (13)$$

Let us rewrite the expressions (12) and (13) as

$$A(s,t) = \sum_{J>1}(J+\tfrac{1}{2})M_{+-}^J(t)P_J''(\cos\varphi), \quad (14)$$

$$B(s,t) + 2m^2A(s,t)$$
$$= \sum_J (J+\tfrac{1}{2})M_{++}^J(t)P_J(\cos\varphi). \quad (15)$$

The projection formulas for M_{+-}^J, M_{++}^J can be worked out and are given by

$$M_{+-}^J(t)$$
$$= [(2J-1)(2J+1)(2J+3)]^{-1}$$
$$\times[(2J+3)A_{J-2} - 2(2J+1)A_J + (2J-1)A_{J+2}], \quad (16)$$

$$M_{++}^J(t) = B_J + 2m^2A_J, \quad (17)$$

where

$$A_J = \int_{-1}^{+1} A(s,\cos\varphi)P_J(\cos\varphi)d(\cos\varphi)$$
$$= \frac{1}{\pi pq}\int ds' A_s(s',t)Q_J[(s'+p^2+q^2)/2pq]$$
$$+ (u\text{-channel contribution}), \quad (18)$$

and a similar expression exists for B_J. In writing (18) we have assumed Mandelstam representation for A and B, and $A_s(s,t)$ is the absorptive part in s channel of the amplitude A.

We thus see from (16)-(18) that the amplitudes $M_{++}^J(t)$ and $M_{+-}^J(t)$ have a Gribov-Froissart representation in terms of the $Q_J(z)$ functions. As such, these objects are the natural objects for which one would make meromorphy assumptions in complex J plane. If we assume that $M_{++}^J(t)$ and $M_{+-}^J(t)$ do not have any fixed singularities in complex J plane, we obtain for the asymptotic behaviors

$$A(s,t) \xrightarrow[s\to\infty]{} \beta_1(t)\frac{\alpha(t)[\alpha(t)-1]}{\sin\pi\alpha(t)}$$
$$\times[1-e^{i\pi\alpha(t)}]s^{\alpha(t)-2}, \quad (19)$$

$$B(s,t) + 2m^2A(s,t) \xrightarrow[s\to\infty]{} \frac{\beta_2(t)[1-e^{-i\pi\alpha(t)}]}{\sin\pi\alpha(t)}s^{\alpha(t)}. \quad (20)$$

Notice that the right-hand side of (19) does not have a pole at $\alpha(t) = 1$.

Now the leading $I = 1$ Regge trajectory is the ρ trajectory. This implies that

$$A(s,t) < (s^{1+\epsilon})^{-1}, \quad \epsilon > 0,$$

for the region $\mathrm{Re}\,\alpha_\rho(t) < 1$, i.e., $t < m_\rho^2$. Hence for $t < m_\rho^2$ the amplitude $A(s,t)$ must satisfy the superconvergence relation

$$\pi^{-1}\int \mathrm{abs.}\,A(s',t)ds' = \pi^{-1}\int a_1(s',t;0,0)ds' = 0 \quad (22)$$
$$(t < m_\rho^2).$$

This obviously contradicts the Fubini sum rule which gives

$$\pi^{-1}\int a_1(s',t;0,0)ds' = F(t). \quad (23)$$

Thus the assumption of pure Regge behavior for the scattering amplitudes is untenable if the Fubini sum rule is right.

We now come to the second part of our investigation and would like to discuss that simplest possible singularity structure consistent with Fubini sum rule. Firstly, in view of the above remark, we would like to have a singularity in the J plane, apart from the ρ Regge trajectory, which would not allow us to write down the inequality (21). A fixed singularity at $J = 1$ would be sufficient for this purpose. A fixed singularity at $J = 1$ is also natural in view of the following circumstance. In the derivation of the Fubini sum rule, we have been treating the electromagnetic interaction only to the second order in e. Under such a scheme of calculation, a fixed $J=1$ singularity, which is physically due to $I=1$, $J=1$ "charged photons," is not surprising. If one included the electromagnetism to all higher orders, this fixed singularity might become a moving singularity by the same mechanism which presumably turns other $J=1$ fixed singularities which occur when one deals with strong interactions also to the lowest order. If this reasoning is correct, we should be able to achieve consistency with a fixed $J=1$ singularity, which does not contribute to the absorptive part of the amplitudes.[3] So one would still be able to maintain pure Regge behavior for absorptive parts. We shall assume that such is the case and explore the consequences. The Fubini sum rule would then be valid only for the range of t given by

$$\mathrm{Re}\alpha_\rho(t)<1, \text{ i.e., } t<m_\rho^2, \tag{24}$$

because otherwise the integral would diverge.

The absorptive behavior of $A(s,t)$, in our model, would be given by

$$A(s,t)\underset{s\to\infty}{\longrightarrow}(\text{a function of } t)/s \quad \text{for } t<m_\rho^2. \tag{25}$$

In fact the Fubini sum rule for the possibility under consideration could be expressed as

$$\lim_{s\to\infty} -sA(s,t)=\lim_{s\to\infty} -\frac{s}{\pi}\int\frac{\mathrm{Im}A(s',t)ds'}{s'-s}=F(t) \tag{26}$$
$$(t<m_\rho^2).$$

In view of (26) we can write

$$A(s,t)\underset{s\to\infty}{\longrightarrow}\frac{-F(t)}{s}+\frac{\bar\beta(t)\{1-\exp[-i\pi\alpha_\rho(t)]\}}{\sin\pi\alpha_\rho(t)} \times s^{\alpha_\rho(t)-2}, \tag{27}$$

where $\bar\beta(t)$ is some new function of t and is not given by Eq. (19), which was written for the case of no fixed singularity. Now we know that $A(s,t)$ has no ρ pole since the partial wave expansion for $A(s,t)$ given by Eq. (14) has no $J=1$ term. Therefore, the function $\bar\beta(t)$ would have to satisfy the consistency condition

$$\lim_{t\to m_\rho^2}\left[-\frac{F(t)}{s}+\frac{\bar\beta(t)\{1-\exp[-i\pi\alpha_\rho(T)]\}}{\sin\pi\alpha_\rho(t)}s^{\alpha_\rho(t)-2}\right]\times(t-m_\rho^2)=0,$$

i.e., $$\{[dF^{-1}(t)/dt]2\bar\beta(t)+\pi d\alpha_\rho(t)/dt\}_{t=m_\rho^2}=0. \tag{28}$$

In conclusion we would like to point out that if one takes the operators $j_\mu^\alpha(x)$ as good interpolating fields for ρ-meson field operators $\rho_\mu^\alpha(x)$, i.e., $\rho_\mu^\alpha(x)=\text{const.}j_\mu^\alpha(x)$, then one would not run into any inconsistency in discussing $\pi+\rho\to\pi+\rho$ amplitudes. These amplitudes would then be given by

$$M_{\mu\nu}^{\alpha\beta}(p_2,K_2;p_1,K_1) =\lim_{K_{1,2}^2\to m_\rho^2}(K_1^2-m_\rho^2)\times(K_2^2-m_\rho^2)T_{\mu\nu}^{\alpha\beta}(p_2,K_2;p_1,K_1)$$

and would satisfy the superconvergence sum rule,[4] instead of Eq. (23), since

$$\lim_{K_{1,2}^2\to m_\rho^2}(K_1^2-m_\rho^2)(K_2^2-m_\rho^2)\times F((K_1-K_2)^2,K_1^2,K_2^2)=0.$$

Thus the present work has no implications about the validity of the Regge-pole hypothesis within the domain of pure strong interactions.

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After completing the work, the author came to know that Professor Low et al. have also arrived at similar conclusions using a different approach.

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[1]S. Fubini, Nuovo Cimento 43A, 475 (1966); S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento 40A, 1171 (1965); D. Amati, R. Jengo, and E. Remiddi, Phys. Letters 22, 674 (1966).

[2]M. A. B. Bég, Phys. Rev. Letters 17, 333 (1966); V. N. Gribov, B. L. Ioffe, and V. M. Shekhter, Phys. Letters 21, 457 (1966).

[3]In a Yang-Mills–type theory these terms would arise from coupling of three vector particles and hence to this order would give no contribute to absorptive part in s channel.

[4]The superconvergence sum rules for $\pi+\rho \rightarrow \pi+\rho$ scattering have been written down by V. DeAlfaro, S. Fubini, G. Rossetti, and G. Furlan, Phys. Letters 21, 576 (1966).

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SIMPLIFIED THEORY OF SYMMETRY CHANGE IN SECOND-ORDER PHASE TRANSITIONS: APPLICATIONS TO V_3Si. Joseph L. Birman [Phys. Rev. Letters 17, 1216 (1966)].

Equation (5) should read

$$A_{1g},\ E_g,\ F_{2g} \text{ not acceptable.} \qquad (5)$$

Table I, right-hand column, line 2, should read

$$A_1 \text{ of } D_4{}^5$$