CURRENT ALGEBRA AND NON-REGGE BEHAVIOR OF WEAK AMPLITUDES

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Certain weak amplitudes exhibit non-Regge-type behavior. These amplitudes have fixed poles in the complex angular-momentum plane which have the dual property of allowing a sum rule of the Dashen–Gell-Mann–Fubini[1,2] type to hold, although one might naively expect a superconvergence relation[3] for this amplitude, and of insuring that spin-one particle poles are reproduced in the left-hand side of the sum rule correctly.

We consider the covariant scattering amplitude $T_{\mu\nu}(s,t,q_1^2,q_2^2)$ for the process

$$\gamma q_{1,\nu}+\pi p_1\rightarrow\gamma q_{2,\mu}+\pi p_2, \tag{1}$$

where $\gamma q_{1,\mu}$ represents an isovector photon of four-momentum $q_{1\lambda}$ and polarization index μ, and π_{p_1} represents a pion of four-momentum $p_{1\lambda}$. The variables s and t are given by

$$s=(p_1+q_1)^2,\quad t=(p_1-p_2)^2.$$

We have supressed isospin indices, since we shall always be dealing with isospin 1 in the t channel $\pi+\pi\rightarrow\gamma+\gamma$. The amplitude $T_{\mu\nu}$ may be expanded in the form

$$T_{\mu\nu}=TP_\mu P_\nu+\cdots, \tag{2}$$

where

$$P_\mu=\tfrac{1}{2}(p_1+p_2)_\mu$$

and the nine omitted covariants are no more than linear in P_μ. The invariant amplitude T has the following properties:

(1) It is odd under $s\longleftrightarrow u$ crossing.

(2) It is proportional to a single helicity amplitude in the t channel, to wit,

$$(\sin^2\theta_t)T\sim f_{1,-1;0,0}(\theta_t)$$

in the notation of Jacob and Wick.[4]

(3) Its absorptive part in the s channel, A, enters into the Dashen–Gell-Mann–Fubini sum rule,[1,2] derived as a consequence of local commutation relations between the time component of the current densities $V_\mu(x)$ coupling to the photons,

$$\int ds' A(s',t,q_1^2,q_2^2)=G(t), \tag{3}$$

where $G(t)$ is the coefficient of P_μ in the vertex $\langle\pi p_2|V_\mu(0)|\pi p_1\rangle$.

The amplitude T possesses a partial-wave expansion in the t channel[5]:

$$T(z_t,t)=\sum_{J=2}^{\infty}e_{20}^{\;J}(z_t)(2J+1)F^J(t), \tag{4}$$

where the partial-wave amplitudes $F^J(t)$ are given for sufficiently large J by the Froissart-Gribov[6] formula

$$F^J(t)=\int^\infty dz_t\,C_{20}^{\;J}(z_t)A(z_t,t). \tag{5}$$

For our purposes, the essential behavior of the e^J and C^J's are

$$e_{20}^{\;J}\propto P_J''/(J-1)^{\frac{1}{2}}, \tag{6a}$$

$$C_{20}^{\;J}\propto(J-1)^{\frac{1}{2}}Q_{J-2}+\cdots. \tag{6b}$$

Thus we are led to the following further properties:

(4) The point $J=1$ corresponds to a sense-nonsense[6] transition in the channel $\gamma+\gamma\rightarrow\pi+\pi$. This refers to the fact that the helicity amplitude $f_{1,-1;0,0}$ is a transition from a two-photon state with total z component of spin equal to 2 in the center of mass and so cannot obtain a contribution from an intermediate state of

spin less than 2. This has the immediate consequence that the amplitude T does not have a pole at $t = m_\rho^2$ arising from an intermediate ρ-meson state.

(5) We may compute the asymptotic behavior of T and A from the exchange of a trajectory in the t channel. We have in F^J a term of the form

$$F^J(t) \propto B(J,t)(J-1)^{\frac{1}{2}}/[J-\alpha(t)], \qquad (7)$$

where we have kept the factor of $(J-1)^{1/2}$ from Eq. (6b) since we will later be particularly interested in the point $J=1$. According to the usual assumptions of Regge theory, $\beta(J,t)$ is nonsingular. Now putting this in Eq. (4) and using the usual Sommerfeld-Watson contour, we obtain the high-s behavior of $T(s,t)$ as

$$T(s,t) \propto \frac{\alpha(t)[\alpha(t)-1]\beta[\alpha(t),t](-s)^{\alpha(t)-2}}{\sin\pi\alpha(t)}. \qquad (8)$$

The factor $\alpha(t)[\alpha(t)-1]$ comes from differentiating the $P_{\alpha(t)}$ in Eq. (6a) and is characteristic of a sense-nonsense transition since it insures that $T(s,t)$ does not have a pole at $t = m_\rho^2$, where $\alpha_\rho(t)=1$. [We have not kept the signature factors explicitly. The ρ trajectory has negative signature and so does contribute to the odd crossing-symmetric amplitude T. The signature factor of $1-e^{-i\pi\alpha(t)}$ does not remove the unphysical pole at $\alpha(t)=1$; the factor $\alpha(t)-1$ is necessary for this.]

We compute the asymptotic part of the absorptive part $A(s,t)$ from Eq. (8),

$$A(s,t) \propto \alpha(t)[\alpha(t)-1]s^{\alpha(t)-2}. \qquad (9)$$

We now wish to establish the range of the variable t for which Eqs. (3), (8), and (9) are valid. Equation (3) is originally derived for t, q_1^2, and q_2^2 spacelike.[1,2] However, it is a relation between analytic functions and can presumably be continued to timelike values of the variables. As soon as we admit this possibility, however, we find certain puzzling features:

(1) The function $G(t)$ has a pole at $t = m_\rho^2$ from ρ exchange. The absorptive part $A(s,t)$ does not have this pole, and it has been suggested by Fubini[2] and Fubini and Segrè[7] that the left-hand side of Eq. (3) develops this pole because the integral diverges when $\alpha(t)=1$ (i.e., at $t = m_\rho^2$) due to the asymptotic form Eq. (9). However, we see that although the integral does indeed diverge, the left-hand side of Eq. (3) does not develop the ρ pole because of the sense-nonsense factor of $\alpha(t)-1$ which cancels the $1/[\alpha(t)-1]$ from the integration. It is not possible to say that the $[\alpha(t)-1]$ factor is not present in Eq. (9), since it is certainly needed in Eq. (8) to prevent the appearance of the unphysical pole at m_ρ^2.

(2) The asymptotic forms of Eqs. (8) and (9) imply that $A(s,t)$ is a superconvergent amplitude. That is, if we write an unsubtracted dispersion relation in s, which should be valid for $\alpha(t)<2$,

$$T(s,t) = \int ds' A(s',t)/(s'-s), \qquad (10)$$

then for $\alpha(t)<1$ we have

$$\lim_{s\to\infty} (-s)T(s,t) = \int ds' A(s',t) = 0, \qquad (11)$$

so that Regge asymptotic behavior coming from the usual moving poles is inconsistent with Eq. (3). In fact, we can easily see what demands the sum rule places on the analytic properties of F^J in the J plane by observing from Eqs. (5) and (6b) that F^J has a pole at $J=1$ arising from the pole in Q_{J-2} when $J-2=-1$. Since the residue of the pole of Q_J at negative integral J is proportional to P_{J+1}, we have

$$\lim_{J\to 1} (J-1)F^J(t) = \int^\infty dz_t A(z_t,t) \sim \int ds' A(s',t,q_1^2,q_2^2). \qquad (12)$$

So the existence of the nonzero right-hand side of Eq. (3) implies the existence of a fixed pole of $F^J(t)$ in the J plane with the vertex $G(t)$ as residue, if the sum rule and Froissart-Gribov continuation are simultaneously valid in a common region of t, while alternatively, the superconvergence relation Eq. (11) is exactly the condition that the residue of the fixed pole vanish.

In order to check whether there is such a pole in the J plane with the proper connection to the vertex, we have considered a simple model of pions interacting strongly with scalar, isoscalar mesons. We shall give the details of the calculation elsewhere and here give only the relevant results. The amplitude $T_{\mu\nu}$

satisfies

$$T_{\mu\nu}(q_1,q_2,p_1,p_2)=I_{\mu\nu}(q_1,q_2,p_1,p_2)+\int\frac{d^4k}{(2\pi)^4}\times\frac{I_{\mu\nu}[q_1,q_2,k+\frac{1}{2}(p_1-p_2),k-\frac{1}{2}(p_1-p_2)]M[\frac{1}{2}(p_2-p_1)-k,\frac{1}{2}(p_1-p_2)-k,p_1,p_2]}{\{[k+\frac{1}{2}(p_1-p_2)]^2-m^2\}\{[k-\frac{1}{2}(p_1-p_2)]^2-m^2\}},\tag{13}$$

where $M(k_1,k_2,p_1,p_2)$ is the π-π scattering amplitude and $I_{\mu\nu}$ is the Bethe-Salpeter irreducible kernel.[8] Projecting the coefficient of $P_\mu P_\nu$ from Eq. (13) and decomposing T and I into partial waves in the t channel according to Eq. (14), we obtain, in the center of mass,

$$F^J(t;\cdots;\cdots)=I^J(t;\cdots;\cdots)+\frac{1}{4\pi^3}\int_{-\infty}^{\infty}dK^0\int_0^\infty K^2dK\left(\frac{K}{P}\right)^2\frac{I^J(t;\cdots;K,K^0)M^J(t;K,K^0;\cdots)}{[-K^2-m^2+(K^0+\frac{1}{2}t^{1/2})^2][-K^2-m^2+(K^0-\frac{1}{2}t^{1/2})^2]},\tag{14}$$

where we have written

$$k_\mu=(\vec{K},K^0),\quad P_\mu=(\vec{P},P^0),$$

and the unspecified variables are the relative momentum and energy in the initial and final states. M^J is the usual t-channel partial-wave amplitude for π-π scattering. Equation (14) may be continued to complex J using Eq. (5), its analog for I^J, and the familair Froissart-Gribov continuation for M^J.

In lowest order

$$I=\frac{4}{s-m^2+i\epsilon}-\frac{4}{u-m^2+i\epsilon},\tag{15}$$

so that in this order I^J has a pole at $J=1$ coming from the pole of Q_{J-2}. We have studied some higher order contributions to I and find they are analytic at $J=1$; we speculate that this is true in general, so that the residue of the pole of the exact kernel at $J=1$ may be computed from Eq. (15). Inserting this residue in Eq. (14), we find

$$\lim_{J\to 1}(J-1)F^J(t;\cdots)\sim 2\left[1+\frac{1}{4\pi^3}\int_{-\infty}^{\infty}dK^0\int_0^\infty K^2dK\left(\frac{K}{P}\right)\times\frac{M^1(t;K,K^0;\cdots)}{[-K^2-m^2+(K^0+t^{1/2}/m)^2][-K^2-m^2+(K^0-t^{1/2}/m)^2]}\right],\tag{16}$$

where we have assumed, of course, that the π-π amplitude has only moving poles (we take the limit with t in a region where M^J is analytic at $J=1$). The right-hand side of Eq. (16) is the Bethe-Salpeter equation (without approximation) for the vertex function $G(t)$, since the matrix element $G_\mu=\langle\pi_{p_2}|V_\mu(0)|\pi_{p_1}\rangle$ satisfies

$$G_\mu(p_2,p_1)=(p_1+p_2)_\mu+\int\frac{d^4k}{(2\pi)^4}\frac{2k_\mu M[p_1,p_2,\frac{1}{2}(p_2-p_1)-k,\frac{1}{2}(p_1-p_2)-k]}{\{[k+\frac{1}{2}(p_1-p_2)]^2-m^2\}\{[k-\frac{1}{2}(p_1-p_2)]^2-m^2\}},\tag{17}$$

and the coefficient of P_μ of this equation, in the center of mass, is precisely the right-hand side of Eq. (16).

So we see that for this model we indeed have a fixed pole of F^J at $J=1$. The residue of this pole is precisely the pion form factor, and the Fubini sum rule, Eq. (3), is an integral expression of this fact. From Eq. (14) we see that F^J also has the moving J-plane poles which contribute to the π-π amplitude.

We now show that the fixed J pole has the following consequences:

(1) It does not lead to a physical pole in T in the t variable at $\alpha(t)=1$.

(2) It does not contribute to the asymptotic behavior of the absorptive part[9] (so that the latter is still proportional to $s^{\alpha(t)-2}$).

(3) It contributes an asymptotic term of the form $G(t)/s$ to the amplitude T.

(4) It allows the left-hand side of Eq. (3) to diverge as one approaches the ρ pole from small t.

We rewrite Eq. (7) as[10]

$$F^J(t) \propto \beta'(J,t)(J-1)^{\frac{1}{2}}/[J-\alpha(t)](J-1), \tag{18}$$

where $\beta'(J,t)$ is nonsingular. We then obtain from Eq. (4)

$$T(s,t) \propto \beta'[\alpha(t),t]\alpha(t)(-s)^{\alpha(t)-2}/\sin\pi\alpha(t) - \beta'(1,t)\alpha(t)(-s)^{-1}/[1-\alpha(t)]\pi. \tag{19}$$

Equation (19) exhibits the asymptotic behavior $1/s$ for $\alpha(t)<1$, and its absorptive part has the form

$$A(s,t) \propto \beta'\alpha(t)(-s)^{\alpha(t)-2}, \tag{20}$$

since the second term in Eq. (19) is real. We note the absence of the factor of $\alpha(t)-1$ in Eq. (20) as compared with Eq. (9). Thus the fixed pole at $J=1$ allows the left-hand side of Eq. (3) to develop a pole[11] for $\alpha(t)=1$, i.e., $t=m_\rho^2$. The reason for this difference is easy to see from Eq. (19). Without the second term, $T(s,t)$ would have a pole at $t=m_\rho^2$ unless there were a factor $\alpha(t)-1$ to remove it. In Eq. (19), however, this pole is removed by the pole of the second term at $t=m_\rho^2$ which has an equal and opposite residue to the Regge pole and is a reflection of the fixed pole at $J=1$. Once the leading trajectory passes to the right of $J=1$, the amplitude will again be Regge-type in the sense that the asymptotic behavior of both T and A will be controlled by $\alpha(t)$. Of course, the sense-nonsense behavior of the absorptive part as $\alpha\to 1$ from above is still anomalous.

Our conclusions are that current-algebra sum rules for weak amplitudes and pure Regge behavior are incompatible if they both are to hold simultaneously in a region of t. Our study of a model for which Eq. (3) is true and which has a continuation in the complex J plane leads us to conclude that these amplitudes have a fixed pole at $J=1$. This pole holds up the asymptotic behavior of the full amplitude (s^{-1} rather than $s^{\alpha(t)-2}$), although the s channel absorptive part has normal Regge asymptotic behavior. This is necessary since we have established that A does not satisfy a superconvergence relation. Finally, the fixed pole does not contribute spurious poles in the t variable to the amplitude since this pole exactly cancels the Regge pole itself at the sense-nonsense point.

It is interesting that the non-Regge behavior we have found applies to an amplitude that is not directly measurable, i.e., scattering of charged photons. Our arguments specifically do not apply to the scattering of real photons, nor in their present form to the photoproduction of e^+,ν or μ^+,ν, since in the latter case there are extra amplitudes present which cannot be analyzed in terms of two-body processes.

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After completion of this paper, we were informed that the importance of the fixed pole at $J=1$ has also been noted by V. Singh.

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[8]We mean by "irreducible kernel" that $I_{\mu\nu}$ cannot be split into two disconnected parts, with the external pion lines in one part and the currents in the other, by cutting two internal pion lines.

[9]It has been observed by K. Bardakci, M. Halpern, and G. Segrè (to be published) that the large-s behavior of T, demanded by Eq. (3), prevents T from Reggeizing. They have speculated that $A(s,t)$ does Reggeize, as we have found.

[10]It is possible that there may be other singularities of F^J in J arising from the integration in Eq. (14). We have verified that no such singularities arise in the ladder approximation to M^J, at least for Re$J>\frac{1}{2}$.

[11]In particular, the asymptotic behavior of Eq. (20) insures that the sum rule is valid until $t=m_\rho^2$.