the K^2 dependence still exists at $\theta = 120°$. Within the limits of error of our experiments, we see that the linewidth appears to be proportional to K^2 up to $\theta = 90°$. However, on closer examination, we note that the point at $\theta = 120°$ lies above the empirical equation $\Gamma = 7.03 \times 10^4 K^2$ with K expressed in Å^{-1}. We now make use of our scattering data from angular dissymmetry measurements.[12] If we take $l^2 = 324\ \text{Å}^2$, $\lambda = 4744\ \text{Å}$, $\tau - 1 = (T - T_c)/T_c = 1.20 \times 10^{-3}$, we find

$$\Gamma = aK^2[1 + (b/a)K^2]$$
$$= 6.32 \times 10^4 K^2(1 + 4.5 \times 10^4 K^2), \qquad (4)$$

where $b/a = l^2/6(\tau - 1)$. Γ and K are expressed in kHz and Å^{-1}, respectively. Equation (4) clearly fits our scattering data. Further work is definitely indicated since we have only one half-width at $\theta = 120°$ showing a measureable deviation from the K^2 dependence. On the other hand, we are certain that Eq. (2), $\Gamma = aK^2$, holds whenever the magnitude of correlation is small and Γ approaches zero as K approaches zero. At intermediate temperature distances, where the effects of correlation may have to be taken into account, we have some evidence to show that Eq. (3) indeed agrees with experiments.

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CONDENSATE TURBULENCE IN A WEAKLY COUPLED BOSON GAS*

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The ring-model approximation is applied to thermal equilibrium below the λ-point, to the relaxation of a chaotic condensate into the zero-momentum state, and to turbulent counterflow of condensate and normal fluid.

This note reports some applications of quantum-field correlation equations whose classical-field limit constitutes a statistical approximation, along the lines of current turbulence theory, for chaotic solutions of the Gross-Pitaevskii equation.[1] The quantum effects prevent an ultraviolet catastrophe.

Assume the standard second-quantized Hamiltonian

$$H = \sum_k k^2 q_k^\dagger q_k + \tfrac{1}{2}\sum_{krs} V_{k-s} q_k^\dagger q_p^\dagger q_r q_s \qquad (1)$$
$$(k + p = r + s),$$

with $V_k = |V_k|$ and $\hbar = 2m = 1$. The correlation equations, called the ring-model approximation[2] (RMA), predict the evolution of $Q_k(t, t') = \langle q_k^\dagger(t') q_k(t)\rangle$ from an initial time. Here $q_k(t)$ is the Heisenberg destruction operator and $\langle\ \rangle$ denotes trace over a Gaussian initial-state ensemble, which can be far from thermal equilibrium. The structure of the approximation is best exhibited by considering statistically steady states and using the spectrum function $Q_k(\omega) = (2\pi)^{-1}\int Q_k(t, t') e^{i\omega(t-t')} d(t-t')$. [All integrations will be over $(-\infty, \infty)$.] After Fourier transformation and some algebraic manipulation, the time-domain RMA equations (Ref. 2, Appendix B) yield

$$2\pi i G_k(\omega) = [k^2+\omega_0-\omega+F_k(\omega)]^{-1}, \tag{2}$$

$$Q_k(\omega) = 2\pi |G_k(\omega)|^2 \sum_{rs} \iint d\omega'' d\omega''' |V_{k-s}(\omega-\omega''')|^2 Q_p^{+}(\omega') Q_r(\omega'') Q_s(\omega'''), \tag{3}$$

where

$$F_k(\omega) = \sum_s \int d\omega' V_{k-s}(\omega-\omega') Q_s(\omega') - 2\pi i \sum_{rs} \iint d\omega'' d\omega''' |V_{k-s}(\omega-\omega''')|^2 Q_p(\omega') Q_r^{+}(\omega'') G_s(\omega''') \tag{4}$$

$$(k+p=r+s, \quad \omega+\omega'=\omega''+\omega'''),$$

$$V_q(\omega) = V_q/[1+V_q W_q(\omega)], \tag{5}$$

$$W_q(\omega) = -2\pi i \sum_p \int d\omega' [(G_p(-\omega'))^* Q_{p+q}(\omega-\omega') - Q_p(-\omega') G_{p+q}(\omega-\omega')], \tag{6}$$

$$Q_k^{+}(\omega) = Q_k(\omega) + B_k(\omega), \quad B_k(\omega) = G_k(\omega) + (G_k(\omega))^*, \tag{7}$$

and $\omega_0 = V_0 N$, N = total number of particles. $G_k(\omega)$ is an admittance function for mean response of $q_k(t)$ to small perturbations, and $V_q(\omega)$ is an effective matrix element. The mean occupancies are $N_k = \int Q_k(\omega) d\omega$, and $\int B_k(\omega) d\omega = 1$. For (2)-(7) to represent a nonequilibrium steady state, external coupling terms must be added which will not be examined explicitly here. The RMA equations also yield

$$\begin{aligned} T_k = 2\pi \sum_{rs} \iiint d\omega d\omega'' d\omega''' |V_{k-s}(\omega-\omega''')|^2 [&B_k(\omega) B_p(\omega') Q_r(\omega'') Q_s(\omega''') - B_r(\omega'') B_s(\omega''') Q_k(\omega) Q_p(\omega') \\ &+ B_k(\omega) Q_p(\omega') Q_r(\omega'') Q_s(\omega''') + B_p(\omega') Q_k(\omega) Q_r(\omega'') Q_s(\omega''') \\ &- B_r(\omega'') Q_k(\omega) Q_p(\omega') Q_s(\omega''') - B_s(\omega''') Q_k(\omega) Q_p(\omega') Q_r(\omega'')] \end{aligned} \tag{8}$$

$$(k+p=r+s, \quad \omega+\omega'=\omega''+\omega'''),$$

where T_k is the net contribution of the interaction to dN_k/dt. In a nonequilibrium steady state, T_k is nonzero and is cancelled by contributions from external couplings.

In thermal equilibrium,[2]

$$Q_k(\omega) = B_k(\omega)/(e^{\beta\omega-\alpha}-1), \tag{9}$$

whence $T_k \equiv 0$. If V is weak and moderate ranged [$\beta_\lambda \omega_0 \ll 1$, $(\beta_\lambda)^{-1} = \lambda$ temperature; $V_k/V_0 \approx 1$ for $\beta_\lambda k^2 < 1$, ≈ 0 for $\beta_\lambda k^2 \gg 1$], (2)-(9) are soluble analytically for $\omega_0^{-1} \gg \beta \gg \beta_\lambda$. They yield $N-N_0 \ll N$, $\alpha \approx \beta\omega_0$. $G_k(\omega_0+\omega)$ couples significantly only to itself and $(G_{-k}(\omega_0-\omega))^*$, and is evaluated as the solution of a complex cubic. The solution gives

$$B_k(\omega_0+\omega) \approx \tfrac{1}{2}\pi^{-1} f(\omega)[(\omega+k^2)/|\omega|](\omega^2-k^4)^{-1/2} \quad (|\omega| > k^2, \ \ k^2 \ll \omega_0),$$

$$B_k(\omega_0+k^2+\omega) \approx \tfrac{1}{2}(\pi\omega_0)^{-1}(-1+4\omega_0/\omega)^{1/2} \quad (0<\omega<4\omega_0, \ \ k^2 \gg \omega_0 \text{ but } V_k \approx V_0), \tag{10}$$

where B_k vanishes for ω outside the stated limits. See Fig. 1 for $f(\omega)$. For $k^2 \to 0$, (10) gives $B_k(\omega_0+\omega) \to \frac{1}{2}\delta(\omega) + \frac{1}{2}\pi^{-1} f(\omega)/\omega$. For $k^2 \ll \omega_0$,

$$|V_k(\omega)|^2 \approx |\omega^2 - k^4|/N^2 \quad (|\omega| \ll \omega_0),$$

$$\approx V_0^2 \quad (|\omega| \gg \omega_0). \tag{11}$$

The correct equation corresponding to (10) is the phonon relation

$$B_k(\omega_0+\omega) \approx [1+(\omega_0/2\omega_k)]\delta(\omega-\omega_k)-(\omega_0/2\omega_k)\delta(\omega+\omega_k), \quad \omega_k = k(2\omega_0)^{1/2} \quad (k^2 \ll \omega_0),$$
$$B_k(\omega_0+k^2+\omega) \approx \delta(\omega-\omega_0) \quad (k^2 \gg \omega_0), \tag{12}$$

which is obtained, among other ways, by examining the response of the Gross-Pitaevskii equation to perturbations of the ground state.[3]

Equation (10) gives a band spectrum whose weights below and above the phonon frequency are such that, used in (9), both (10) and (12) give $N_k \approx (2\beta k^2)^{-1}$ $(k^2 \ll \omega_0)$, which is $\frac{1}{2}$ the free-boson value.[4] For $k^2 \gg \omega_0$, both give $\int \omega B_k(\omega) d\omega \approx k^2+2\omega_0$. Both give $U_0 \approx \omega_0 N/2$, $E_k \approx \beta^{-1}$ $(k^2 \ll \omega_0)$, $E_k \approx (k^2+\omega_0)/\{\exp[\beta(k^2+\omega_0)]-1\}$ $(k^2 \gg \omega_0)$, where the total energy is $U_0 + \sum_k E_k$.[5] This U_0 equals the classical-field limit of the lower bound demonstrated for the eigenvalues of the RMA model Hamiltonian.[2]

If Q^+ is replaced by Q in (2)-(7) and the two $BBQQ$ terms are omitted in (8), the results are the RMA for the Gross-Pitaevskii equation. In this classical-field limit, (9) degenerates to $Q_k(\omega) = B_k(\omega)/(\beta\omega-\alpha)$ and there is an ultraviolet catastrophe, which is independent of the statistical approximation. Although the classical Gross-Pitaevskii equation gives $B_k(\omega)$ correctly, it gives an inadmissible equilibrium $Q_k(\omega)$ for $\beta k^2 > 1$.

The cited results indicate satisfactory thermodynamic predictions for RMA in the β range considered. Also, RMA gives expressions[2] for the density components ρ_k, and it is found that the predicted suppression of density fluctuations below free-particle levels has the correct strength and k dependence. These features provide some motivation for trying the approximation in nonequilibrium situations where $N_k > 1$ for $k^2 < \omega_0$.

Serious deficiencies in equilibrium show up

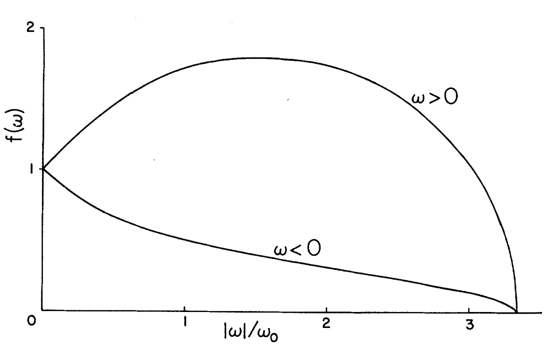

FIG. 1. The function $f(\omega)$ which appears in Eq. (10).

for $\beta\omega_0 \gg 1$. The predicted quantum depression of ground-state energy below $\omega_0 N/2$ has the correct magnitude, but the difference between (10) and (12) is now thermodynamically significant. N_k is not given correctly for $\beta^{-1} < k^2 < \omega_0$, and the specific heat is $\propto \beta^{-3}$ instead of $\propto \beta^{-3/2}$.

The behavior out of equilibrium can be inferred from (8), whose right-hand side is a sum of input and output terms from collisions, thereby resembling the spectral-transfer expression in analogous approximations for Navier-Stokes turbulence.[6] If N_k, N_p, N_r, N_s are all $\ll 1$, the $BBQQ$ terms dominate and the right-hand side looks like a Boltzmann collision integral. If the N's are $\gg 1$, the $BQQQ$ terms dominate, and the extra factor Q implies enhanced transition rates.

Consider the supercooled initial state $N_k = \exp(\alpha_0 - \beta_0 k^2)$, where α_0 and β_0 are chosen to give an eventual equilibrium $N - N_0 \ll N$, $\omega_0^{-1} \gg \beta \gg \beta_\lambda$. Initially, most particles have $k^2 \gg \omega_0$, and $V_k(\omega)$ and $B_k(\omega)$ are replaceable in (8) by V_k and $\delta(\omega - k^2)$. The $BBQQ$ terms cancel, but the $BQQQ$ terms increase occupancies at low and high k at the expense of intermediate k. This leads to a first stage of condensation where a number of particles approximating the eventual equilibrium N_0 have concentrated into a region $k \lesssim k_*$ with $\omega_0 \ll k_*^2 \ll \beta^{-1}$. In this stage, k_* shrinks with a halving time $\sim k_*^2/\omega_0^2$ and the shrinking is dominated by processes local in momentum $(k, p, r, s \sim k_*)$. The N_k for $k \gg k_*$ are very nearly in thermal equilibrium among themselves.

After $k_*^2 = \omega_0$ is reached, the deviation of $V_k(\omega)$ and $B_k(\omega)$ from free-particle values becomes important. Although $V_k(\omega)$ has not its equilibrium form, the essential behavior is indicated by (11). The high k continue to be nearly in internal equilibrium, but at α and β slightly different from the eventual values and such that net scattering out of $k \lesssim k_*$ from collisions with the high-k particles is very small. The shrinking remains dominated by local interactions $k, p, r, s \sim k_*$. For these, $\omega \sim k_*^2$ typically, and $|V_k(\omega)| \sim \omega/N$, which is independent of V_0. Consequently, the halving time for k_* is now $\sim k_*^{-2}$. There is a transition region $k > k_*$, analogous to the inertial range in Navier-Stokes turbulence, through which the kinetic energy squeezed from the collapsing condensate is passed up to high k. Analysis of (8) gives $N_k \propto k^{-13/3}$ here, provided k_* is small enough that $k^2 < \omega_0$ throughout the range. After $k_*^2 \sim \omega_0$ is passed, the total time for attaining the equilibrium condensate ($N_0 \sim N$) is $\sim L^2$ ($\sim mL^2/h$ in ordinary units), where L is the cyclic-box side.

The maximum propagation speed of condensate phase under the exact Hamiltonian is plausibly the phonon speed. But that is not directly relevant to relaxation into equilibrium. What counts is the rate at which phase disturbances dissipate. The RMA halving time k_*^{-2} for $k_*^2 \ll \omega_0$ is the typical time both for evolution of incompressible Navier-Stokes turbulence of velocity k_* and eddy size k_*^{-1} and for nonlinear distortion of a sound wave of wave number k_* and particle velocity k_*. The behavior of $V_k(\omega)$ at low k can be shown to be associated with suppression of density fluctuations in the chaotic condensate, and its independence on V_0 corresponds to the independence of low-Mach-number nonlinear effects in a Navier-Stokes fluid on the value of the compressibility.

If a dilute gas of foreign particles is coupled to the bosons with an infinitesimal potential ϵV_k, RMA gives an effective matrix element $\epsilon|V_q(\omega)|$ for scattering with momentum transfer q and energy transfer ω. Equation (11) then indicates anomalously low cross sections for small q and ω, and suggests anomalously low mutual friction effects.

A crude model of turbulent counterflow is obtained by interrupting the condensation process when $k_*^2 \ll \omega_0$ and displacing the normal fluid ($k > k_*$) by a drift momentum $q \ll \omega_0^{1/2}$; i.e., $N_k \to N_{k+q}$ for $k > k_*$. The immediate effect found from (8) is a rapid loss of particles from $k < k_*$ to $r, s \sim \omega_0^{1/2}$. However, the region $k \gg q$ quickly adjusts to an altered state of near internal equilibrium, as in the isotropic condensation, so that this loss becomes small.[7] The indicated steady state (with appropriate external couplings to maintain the drift) has a skewed condensate with $k_* \sim q$ and a particle exchange between $k < k_*$ and $k > k_*$ which is dominated by local interactions and characterized by the turnover time k_*^{-2}. The typical unbalance in momentum transfer across the skewed condensate is $\sim q$, and the net rate of momentum exchange between $k < k_*$ and $k > k_*$ (mutual friction) is therefore $\sim q^3$ per condensate particle [$\sim q^3/(mh)$ in ordinary units].

The relation to the similar mutual friction found by Vinen[8] for He II is uncertain. The true behavior of a weakly coupled gas, in which the kinetic energy of a typical normal-fluid particle greatly exceeds the energy penalty for knocking a condensate particle out of a correlated state, may be quite different from He II. Moreover, the excitations predicted by RMA are specified only by spectra, and it is fanciful to think of them as vortices or other definite structures whose form implies highly organized phase relations among many wave vectors. But it is significant that a statistical approximation at this level of crudity yields an anomalously low mutual friction. The RMA equations for $L = \infty$ do not appear to give a critical velocity below which mutual friction disappears.

The mean condensate amplitude $\langle\psi(x)\rangle$ is zero in all the analysis reported above; the initial ensembles are Gaussian and all eventual phases are equally probable. RMA can be extended to nonzero-mean ensembles in the way that analogous statistical equations for Navier-Stokes turbulence are extensible to flows with mean velocities.[6] The resulting equation for $\langle\psi(x)\rangle$ is the Gross-Pitaevskii equation augmented by terms expressing coupling to the second-quantized fluctuations, the latter being specified by Q and G functions. These terms are somewhat analogous to the viscous terms in the Navier-Stokes equation and prevent an ultraviolet catastrophe. An essential prerequisite to the extension is that the zero-mean RMA equations do give the correct classical-field ground-state energy $\frac{1}{2}\omega_0 N$. Applied to the relaxation of a chaotic condensate, or to turbulent counterflow, the difference between zero-mean and nonzero-mean equations is like that between treating the velocity field by statistical approximation in homogeneous Navier-Stokes turbulence and integrating the Navier-Stokes equation forward in time point-to-point from a turbulent initial condition. The latter procedure is more accurate but can require enormously more computation. The nonzero-mean equations appear to give correct equilibrium thermodynamics for $\beta\omega_0 \gg 1$ and give correct phonon frequencies. The zero-mean equations may be useful principally in describing condensate turbulence, where faithful reproduction of phonon behavior is unimportant.

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[4]N_k is not the number of quasiparticles of momentum k. One quasiparticle implies $N_k \gg 1$, $N_k - N_{-k} = 1$.

[5]With care taken to include in E_k the excitation effects on $Q_0(\omega)$, (9) and the Hugenholtz-Pines formula yield $E_k \approx \int \omega Q_k(\omega_0+\omega)d\omega$. The quantum correction to U_0 is $\ll U_0$, and the quantum corrections to N_k and E_k for low-lying k are negligible if $\beta\omega_0 \ll 1$.

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EXPERIMENTAL TESTS OF THE CRITICAL-STATE MODEL FOR HYSTERETIC SUPERCONDUCTORS*

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New types of experiments which provide direct checks of the critical-state model of hysteresis in type-II superconductors are described. The observed behavior cannot be accounted for by the critical-state concept.

Discussions of the hysteretic behavior of type-II superconductors are usually based on the critical-state model. An example of the way in which calculations of the magnetization curves are made using this model is given in the paper of Fietz et al.[1] (We also refer the reader to this paper for references to the work of Bean and Kim and their co-workers.) In the simplest version of the model a maximum or critical current of constant magnitude J_c flows everywhere in the sample upon removal of a field greater than H_{c2}. In practice, it has always been necessary in studies such as those of Ref. 1 to assume some functional dependence of J_c on the internal field B in order to deduce the observed magnetization curves. This procedure appears to us to preclude a direct experimental check of the critical-state hypothesis, which is assumed a priori as the basis of the calculation of $J_c(B)$. In this paper we wish to describe a number of new experimental measurements which do provide such a direct check on the critical-state model. The results presented here concern the behavior of the remanent magnetic moment or trapped flux in zero external field with temperature and transport current. We shall also summarize how the magnetization in a field changes in the presence of applied transport currents. We shall refer to the zero-field remanent magnetization as tf.

Flux penetration of a long sample in a parallel external field H begins when $H=H_{c1}$. Starting with an initially unmagnetized sample at some temperature $T_1 < T_c$, the application and removal of any field greater than H_{c1} results in a corresponding tf. The quantity of tf increases as the field is increased from H_{c1} to a value H' (which corresponds to $2H^*$ in the Bean model[2]). At this field there is no further increase in tf. (There often is a subsequent decrease in tf for $H \geq H'$, which we shall ignore in the discussion of this paper. This point will be discussed in a future paper.) Magnetization cycles in which a field less than H' is applied and removed we shall call minor hysteresis loops, and cycles corresponding to $H \geq H'$, the major hysteresis loop. A basic feature of the critical-state model is the assertion that the local macroscopic current density is everywhere equal to $\pm J_c$, or 0. Thus, the tf resulting after cycling through a minor hysteresis loop ($\mathrm{tf}_{\mathrm{minor}}$) is associated, according to this model, with an inner core in which no current