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DYNAMICAL APPROACH TO CURRENT ALGEBRA

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(Received 12 December 1966)

An effective Lagrangian for soft-pion interactions is constructed such that lowest order perturbation theory precisely reproduces the results of current algebra.

In the last year we have seen the development of a "current-algebra method" for calculating soft-pion matrix elements by direct use of partially conserved axial-vector current (PCAC) and current commutation relations.[1,2] However, despite its successes, this method suffers from some serious deficiencies:

(1) The algebraic effort required goes up rapidly with the number of soft pions.[3] This is of more than academic importance, because we would like to be able to add up the emission rates for arbitrary numbers of soft pions, as we already can do for photons and gravitons.

(2) A related problem is that the current-algebra method yields matrix elements which, although symmetric in the soft-pion labels, are not explicitly so.[4]

(3) In manipulating time-ordered products of currents, we lose sight of the dynamics underlying the results obtained. This is good, in that we are freed from dependence on specific dynamical assumptions, but also bad, in that we are offered no clue on how to go beyond the soft-pion limit.

This note will present a new technique for doing soft-pion calculations, which is guaranteed to give the same results as the current-algebra method, but which also is (1) vastly easier, (2) manifestly in accord with Bose statistics, (3) naturally suggestive of dynamical models which can carry us far beyond the results of current algebra.

First, let us recall the structure of the formulas provided by the current-algebra method. The S-matrix element for emission and absorption of any number of soft pions in a reaction $\alpha\to\beta$ is given as a sum matrix of elements between states β and α of a time-ordered product of a number of vector and axial-vector currents times a factor $(G/2m_N)(g_V/g_A)$ for each soft pion.[5] These currents are to be taken at small momenta; hence their matrix element is evaluated by hooking them on to the external lines of the process $\alpha\to\beta$ (as in the case of inner bremstrahlung), and we pick up an extra factor $(g_V/g_A)_n^{-1}$ for each axial current attached to the nth external line.

The key point in the above is that the soft-pion matrix element is of the lowest possible order in G, except for higher order terms in the factors g_V/g_A and in the matrix element $M_{\beta\alpha}$ for the process without soft pions. <u>Hence we will necessarily get precisely the results of current algebra if we evaluate the soft-pion matrix elements to lowest order in G and then insert the correct values of g_V/g_A and $M_{\beta\alpha}$.</u> To reiterate in more detail, the procedure to be followed is listed below:

(a) Choose any Lagrangian which satisfies PCAC and the proper current commutation relations.

(b) Evaluate the desired soft-pion matrix element to lowest order[6] in G.

(c) Write the result in the form dictated by current algebra, i.e., "trees" of soft pions attached to vector and axial-vector vertices on the <u>external</u> lines of a "core" process $\alpha\to\beta$. We will construct below a Lagrangian which directly yields soft-pion matrix elements of this form, so that this step does not require any further effort.

(d) Supply higher order corrections by multiplying with factors g_V/g_A for each soft pion and g_A/g_{Vn} for each axial-vector vertex on the nth external line, and by using the exact value of the matrix element $M_{\beta\alpha}$ for the process without soft pions. In the simplest cases, like $\pi+N\to\pi+N$ or $2\pi+N$ near threshold, the transition $\alpha\to\beta$ is a trival one-particle transition and no $M_{\beta\alpha}$ is needed.

The above procedure will be referred to as the chiral-dynamics method. It does not supplant current algebra, since for the present the only justification for the chiral-dynamics method is that its results agree with those of the current-algebra method, but as a computational technique it is far simpler and more transparent.

We will now proceed to the construction of a Lagrangian which in lowest order immediately reproduces the results of the current-algebra method. The time-honored example of a Lagrangian satisfying PCAC and the chiral commutation relations is that of the σ model[7]:

$$\begin{aligned}\mathcal{L} = &-\bar{N}[\gamma^\mu\partial_\mu + m_{N0} - G_0(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)]N\\ &-\tfrac{1}{2}[\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} + m_{\pi0}^2\vec{\pi}^2] - \tfrac{1}{2}[\partial_\mu\sigma\partial^\mu\sigma + m_{\sigma0}^2\sigma^2]\\ &-(m_{\sigma0}^2 - m_{\pi0}^2)[(G_0^2/8m_{N0}^2)(\vec{\pi}^2+\sigma^2)^2 - (G_0/2m_{N0})(\vec{\pi}^2+\sigma^2)\sigma].\end{aligned} \tag{1}$$

However, although not incorrect from the standpoint of current algebra, this Lagrangian is not well suited to the purposes of the chiral-dynamics method,[8] because even in lowest order it does not yield matrix elements which are manifestly of the "current-algebra form" required by rule (c) above. For instance, the lowest order σ model yields a low-energy πN scattering matrix equal to the sum of poles for one-nucleon exchange (using pseudoscalar coupling) in the s and u channels, plus one σ exchange in the t channel. It is an exercise in Dirac algebra to show that this result is actually equal to what we expect from current algebra, i.e., a sum of poles for one-nucleon exchange (using pseudovector coupling) in the s and u channels plus an equal-time commutator term which looks like one ρ exchange in the t channel. Rule (d) tells us then to multiply this last term by $(g_V/g_A)^2$, and we emerge with precisely the result[9] of the current-algebra method. A similar calculation is known to work for π-π scattering.[2] More generally, if we wished to use the σ model to calculate soft-pion emission and absorption in a physical process $\alpha\to\beta$ more complicated than a single-particle transition, we should have to add graphs in which pions are emitted from internal as well as external lines of this process, and it would take a major effort to rewrite the result in the current-algebra form of rule (c).

The trouble with the σ model stems from its nonderivative $\bar{N}\gamma_5\vec{\tau}\cdot\vec{\pi}N$ and $\sigma\vec{\pi}^2$ interactions; therefore, let us transform them away. The Lagrangian (1) is invariant (except for the $m_{\pi0}^2$ terms) under a chiral SU(2)⊗SU(2) group, under which $\vec{\pi}$ and σ-m_{N0}/G_0 transform as a four-vector. Hence at every point we may define a new nucleon field N' by a chiral transformation

$$N = (1+\vec{\xi}^2)^{-1/2}(1+i\gamma_5\vec{\tau}\cdot\vec{\xi})N', \tag{2}$$

with $\vec{\xi}$ chosen so that

$$\bar{N}[m_{N0} - G_0(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)]N = \bar{N}'[m_{N0} - G_0\sigma']N', \tag{3}$$

$$m_{N0} - G_0\sigma' \equiv [(m_{N0}-G_0\sigma)^2 + G_0^2\vec{\pi}^2]^{1/2}. \tag{4}$$

The required $\vec{\xi}$ is

$$\vec{\xi} = G_0\vec{\pi}[m_{N0} - G_0\sigma + \{(m_{N0}-G_0\sigma)^2 + G_0^2\vec{\pi}^2\}^{1/2}]^{-1}. \tag{5}$$

Expressions like (4) and (5) are, of course, to be interpreted as power series in G_0.

The transformation (2) replaces the $\vec{\pi}$ dependence of $\mathcal{L}$ with a $\vec{\xi}$ dependence, arising from the presence both of $m_{\pi0}^2$ terms which break the chiral symmetry, and of derivative terms which are only invariant under constant chiral transformations. We will therefore define a new pion field $\vec{\pi}'$ as a function of $\vec{\xi}$. Different choices of this function will yield different matrix elements off the pion mass shell, but all give the same S matrix, so that we can define $\vec{\pi}'$ almost as we like. One choice would be $\vec{\pi}' \equiv \vec{\pi}$; $\mathcal{L}$ would then reproduce the results of the current-algebra method both on and off the mass shell, but would be quite complicated. We shall instead make the much more convenient choice

$$\vec{\pi}' \equiv 2m_{N0}\vec{\xi}/G_0. \tag{6}$$

A straightforward calculation shows that the Lagrangian (1) may be written in terms of the new field as

$$\mathcal{L} = -\bar{N}'\left[\gamma^\mu\partial_\mu + m_{N0} - G_0\sigma' + i\gamma^\mu\left(1+\frac{G_0^2\vec{\pi}'^2}{4m_{N0}^2}\right)^{-1}\left\{\left(\frac{G_0}{2m_{N0}}\right)\gamma_5\vec{\tau}\cdot\partial_\mu\vec{\pi}' + \left(\frac{G_0^2}{4m_{N0}^2}\right)\vec{\tau}\cdot\vec{\pi}'\times\partial_\mu\vec{\pi}'\right\}\right]N'$$

$$-\tfrac{1}{2}[\partial_\mu\sigma'\partial^\mu\sigma' + m_{\sigma0}^2\sigma'^2] - (m_{\sigma0}^2 - m_{\pi0}^2)\left[\left(\frac{G_0^2}{8m_{N0}^2}\right)\sigma'^4 - \left(\frac{G_0}{2m_{N0}}\right)\sigma'^3\right]$$

$$-\frac{1}{2}\left(1-\frac{G_0\sigma'}{m_{N0}}\right)\left[\left(1+\frac{G_0^2\vec{\pi}'^2}{4m_{N0}^2}\right)^{-2}\partial_\mu\vec{\pi}'\partial^\mu\vec{\pi}' + \left(1+\frac{G_0^2\vec{\pi}'^2}{4m_{N0}^2}\right)^{-1}m_{\pi0}^2\vec{\pi}'^2\right]. \tag{7}$$

The Lagrangian $\mathcal{L}$ satisfies the requirements of current algebra for all values of $m_{\sigma0}$; so we are free to take $m_{\sigma0}$ as large as we like. But inspection of (7) shows that all graphs containing internal σ' lines vanish as $m_{\sigma0}\to\infty$, and so the σ' field may be dropped everywhere in (7), yielding as our model

$$\mathcal{L}' = -\bar{N}'\left[\gamma^\mu\partial_\mu + m_{N0} + i\gamma^\mu\left(1+\frac{G_0^2\vec{\pi}'^2}{4m_{N0}^2}\right)^{-1}\left\{\left(\frac{G_0}{2m_{N0}}\right)\gamma_5\vec{\tau}\cdot\partial_\mu\vec{\pi}' + \left(\frac{G_0^2}{4m_{N0}^2}\right)\vec{\tau}\cdot\vec{\pi}'\times\partial_\mu\vec{\pi}'\right\}\right]N'$$

$$-\frac{1}{2}\left(1+\frac{G_0^2\vec{\pi}'^2}{4m_{N0}^2}\right)^{-2}\partial_\mu\vec{\pi}'\partial^\mu\vec{\pi}' - \frac{1}{2}\left(1+\frac{G_0^2\vec{\pi}'^2}{4m_{N0}^2}\right)^{-1}m_{\pi0}^2\vec{\pi}'^2. \tag{8}$$

This last step is only permissible because σ' is a chiral invariant and therefore plays no role in maintaining the chiral invariance of $\mathcal{L}$; in contrast, σ is not a chiral invariant, and graphs generated by (1) which contain internal σ lines do not vanish as $m_{\sigma0}^2\to\infty$.

Inspection of Eq. (8) shows immediately that its lowest order graphs are automatically of the current-algebra form required by rule (c), i.e., soft pions hang in clusters from vector and axial-vector vertices on the external lines of our process, the derivatives in (8) preventing soft pions from arising from internal lines. We are to use (8) to lowest order in G, not G_0; so G will appear in place of G_0, m_N in place of m_{N0}, etc. Further, rule (d) tells us that each soft pion introduces an extra factor g_V/g_A, and each $\gamma_5\gamma_\mu$ is accompanied with an extra factor g_A/g_V. Thus the prescription of the chiral-dynamics method for calculating soft-pion emission and absorption in a reaction $\alpha\to\beta$ may now be summarized as follows: Calculate the desired matrix element to lowest order in G, treating $M_{\beta\alpha}$ as a known black box of zeroth order, and using for the soft pions the effective Lagrangian[10]

$$\mathcal{L}_{\text{eff}} = -\bar{N}\left[\gamma^\mu\partial_\mu + m_N + i\gamma^\mu\left\{1+\frac{G^2}{4m_N^2}\left(\frac{g_V}{g_A}\right)^2\vec{\pi}^2\right\}^{-1}\left\{\left(\frac{G}{2m_N}\right)\gamma_5\vec{\tau}\cdot\partial_\mu\vec{\pi} + \left(\frac{G^2}{4m_N^2}\right)\left(\frac{g_V}{g_A}\right)^2\vec{\tau}\cdot\vec{\pi}\times\partial_\mu\vec{\pi}\right\}\right]N$$

$$-\frac{1}{2}\left\{1+\frac{G^2}{4m_N^2}\left(\frac{g_V}{g_A}\right)^2\vec{\pi}^2\right\}^{-2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - \frac{1}{2}\left\{1+\frac{G^2}{4m_N^2}\left(\frac{g_V}{g_A}\right)^2\vec{\pi}^2\right\}^{-1}m_\pi^2\vec{\pi}^2. \tag{9}$$

For instance, we now get the correct[9] πN scattering lengths by just looking at the $\vec{\pi}\times\partial_\mu\vec{\pi}$ term in

Eq. (9); this is surely the simplest derivation of the Adler-Weisberger relation! Also, by expanding the denominators in the last two terms of (9), we immediately get an effective Lagrangian for pion-pion scattering,

$$\mathcal{L}_{\pi\pi}=\frac{G^2}{8m_N^{\,2}}\left(\frac{g_V}{g_A}\right)^2\vec{\pi}^2[2\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}+m_\pi^{\,2}\vec{\pi}^2]. \quad (10)$$

This gives the same π-π matrix element (on the mass shell) as does current algebra,[2] a point essential to the consistency of our method since (9) generates peripheral π-π scattering graphs which make important contributions to processes like $\pi+N\rightarrow 2\pi+N$.

There is no difficulty in including K mesons and hyperons in our model Lagrangian, or in using it to treat weak and electromagnetic as well as strong interactions. A more intriguing extension of this approach would be to start by adding Yang-Mills vector and axial-vector fields in (1), in such a way that it is only their bare mass, and that of the pion, which break chiral invariance; the $\vec{\pi}\times\partial_\mu\vec{\pi}$ term in the transformed Lagrangian then comes entirely from ρ exchange. This offers hope of reproducing the results of current algebra from a Lagrangian which at the same time is a reasonable phenomenological model of the strong interactions. A paper on this model is in preparation.

Finally, it is remarkable that the Lagrangian (9) is to be used only in lowest order,[11] the effects of loops, etc., being already accounted for by the presence in (9) of the factors g_V/g_A. Just how does this come about? If we knew, we might understand more deeply the dynamical basis of current algebra.

I am very grateful for the hospitality extended to me by the Physics Department of Harvard University. I would also like to thank R. Dashen and F. J. Dyson for interesting discussions.

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[1]For a general review of soft-pion calculations see the rapporteur's talk by R. F. Dashen, in Proceedings of the Thirteenth International Conference on High Energy Physics, Berkeley, California, 1966 (unpublished).

[2]The author's previous approach to soft-pion problems is described in Phys. Rev. Letters 17, 616 (1966).

[3]The most thorough statement of the results of current algebra and PCAC for arbitrary numbers of soft pions is that of H. Abarbanel and S. Nussinov, to be published. This article shows how complicated the current-algebra method gets for three or more soft pions.

[4]For instance, see Ref. 3.

[5]Here G is the renormalized pion-nucleon coupling constant, m_N is the nucleon mass, and g_V and g_A are the renormalized vector and axial-vector coupling constants of the nucleon; these factors of course arise from PCAC and the Goldberger-Treiman relation. The currents are defined here not to include the factor g_V, so their commutation relations do not involve any coupling constants.

[6]All unrenormalized pion coupling constants are here to be regarded as power series in the corresponding renormalized coupling constants (of which we ultimately keep only the lowest order terms), and all renormalized pion coupling constants are to be expressed in terms of the pion-nucleon coupling constant G by using Goldberger-Treiman relations.

[7]J. Schwinger, Ann. Phys. (N. Y.) 2, 407 (1957); M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960). The label 0 means "unrenormalized."

[8]The σ model is also useless as a phenomological model of strong interactions; for were we to take it seriously, we should have to give the σ a width $\Gamma_\sigma \cong (3G^2/32\pi)(m_\sigma^{\,3}/m_N^{\,2})(g_V/g_A)^2$, which is larger than m_σ if $m_\sigma > 500$ MeV. Such a broad S-wave resonance seems unlikely.

[9]See, e. g., Ref. 2, Eq. (6), or the other references quoted therein.

[10]For processes involving a single soft pion this is just the pseudovector coupling model, a result already known from current algebra; see S. Weinberg, Phys. Rev. Letters 16, 879 (1966). It is of course an old story that there is some sort of equivalence between pseudovector and pseudoscalar coupling, as shown by F. J. Dyson, Phys. Rev. 73, 929 (1948); L. L. Foldy, Phys. Rev. 84, 168 (1951); etc. The complications encountered in these early papers arose because they dealt with theories having no sort of chiral invariance.

[11]In this respect our method is reminiscent of the recently developed "source" approach of J. Schwinger. I wish to thank Professor Schwinger for an interesting discussion on this point.