where s_0 is a scaling factor which is undetermined in the Regge theory. The two common choices of s_0 are $s_0=\text{constant}$[10] or $s_0=2m_A m_B$.[11] The former choice combined with universality leads to sum rules in which all cross sections are taken at the same s value. The latter choice of s_0 (plus universality) is precisely equivalent to our relative-velocity prescription. It implies that the Regge trajectories are associated primarily with the quark-antiquark systems rather than the hadron-antihadron systems. Of course, the value of s_0 is irrelevant unless one has a theory (quark decomposition or universality) which relates the residues β_i.

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ANGULAR MOMENTUM UNCERTAINTY RELATION AND THE THREE-DIMENSIONAL OSCILLATOR IN THE COHERENT STATES

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An angular momentum uncertainty relation is obtained in terms of sine and cosine operators that have a meaning even in a second-quantization formalism. For a three-dimensional oscillator in coherent states the new uncertainty product is a minimum for large ΔL_z. Even for small ΔL_z the uncertainty product is very small.

It has long been known[1] that the interpretation of the commonly accepted uncertainty relation between angular momentum and angle,

$$\Delta L_z \Delta\varphi \geq \tfrac{1}{2}\hbar, \tag{1}$$

is not precise. The relation lacks meaning for small values of ΔL_z since the angle φ is restricted to values of $(0,2\pi)$. In recent studies, Judge[2] and Susskind and Glogower[3] have independently suggested that the angle variable ψ defined by

$$\psi \equiv \varphi(\text{mod}2\pi) \tag{2}$$

be used so that

$$[\psi, L_z]=i\hbar\left[1-2\pi\sum_{n=-\infty}^{\infty}\delta(\varphi-[2n+1]\pi)\right]. \tag{3}$$

Although an uncertainty relation can be defined by the commutator of L_z with ψ,[2,4] ψ lacks a well-defined operator definition and continuous eigenspectrum.

In this note we propose an alternative uncertainty relation to either (1) or that obtained from Eq. (3). It is defined in terms of sine and cosine operators that have a meaning even in a second-quantization formalism. We evaluate the new expression for a three-dimensional oscillator in coherent states. It is found that for large ΔL_z the coherent states are minimum-uncertainty states in angular momentum and angle, as well as minimum-uncertainty states in position and momentum[5] and in number and phase.[6] Further, even for small ΔL_z the uncertainty product is very small.

Well-defined angle operators can be constructed if we consider the sine and cosine of φ. This is suggested by the S and C operators recently considered by Carruthers and Nieto[6] (hereafter called CN) in studying the number-phase uncertainty relation. Setting $\hbar = 1$ and using the definitions

$$L_z = (\vec{r}\times\vec{p})_z = \frac{1}{i}\frac{\partial}{\partial\varphi},$$
$$\sin\varphi = y/(x^2+y^2)^{1/2},$$
$$\cos\varphi = x/(x^2+y^2)^{1/2}, \tag{4}$$

one easily obtains[7]

$$[\sin\varphi, L_z] = i\cos\varphi, \tag{5}$$

$$[\cos\varphi, L_z] = -i\sin\varphi. \tag{6}$$

From this we deduce the uncertainty relations

$$(\Delta L_z)^2(\Delta\sin\varphi)^2 \geqslant \tfrac{1}{4}\langle\cos\varphi\rangle^2, \tag{7}$$

$$(\Delta L_z)^2(\Delta\cos\varphi)^2 \geqslant \tfrac{1}{4}\langle\sin\varphi\rangle^2, \tag{8}$$

where

$$(\Delta x)^2 \equiv \langle x^2\rangle - \langle x\rangle^2. \tag{9}$$

Note that Eq. (7) becomes the standard form (1) when ΔL_z is large, i.e., φ is small.

We will consider a three-dimensional isotropic harmonic oscillator. (The results for the nonisotropic case are obtained by a straightforward generalization.) The coherent states, which are solutions for the oscillator along any axis, have the properties that

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle, \tag{10}$$

$$|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}|n\rangle$$
$$\equiv A(\alpha)\,|0\rangle, \tag{11}$$

$$\langle\alpha|N_{\text{op}}|\alpha\rangle = |\alpha|^2, \tag{12}$$

where α is any complex number. Let $|\alpha\rangle$ and $|\beta\rangle$ be the coherent states quantized along the x and y axes, respectively, and $a^\dagger$ and $b^\dagger$ be the creation operators acting on the $|\alpha\rangle$ and $|\beta\rangle$ states. Then if ω is the oscillator frequency and d the mean square position, so that

$$x = d(a+a^\dagger),\quad p_x = -im\omega d(a-a^\dagger),$$
$$y = d(b+b^\dagger),\quad p_y = -im\omega d(b-b^\dagger), \tag{13}$$

we have

$$\langle L_z\rangle = \langle\beta,\alpha|(xp_y - yp_x)|\alpha,\beta\rangle$$
$$= 2[(\mathrm{Re}\alpha)(\mathrm{Im}\beta) - (\mathrm{Re}\beta)(\mathrm{Im}\alpha)]. \tag{14}$$

By then calculating $\langle L_z^2\rangle$ and using Eq. (9), we find

$$(\Delta L_z)^2 = |\alpha|^2 + |\beta|^2 = N_x + N_y \equiv N. \tag{15}$$

$\langle\sin\varphi\rangle$ is given by

$$\langle\sin\varphi\rangle = \left\langle\beta,\alpha\left|\frac{b+b^\dagger}{[(a+a^\dagger)^2+(b+b^\dagger)^2]^{1/2}}\right|\alpha,\beta\right\rangle. \tag{16}$$

Using the relations[8]

$$A^\dagger(\alpha)A(\alpha) = 1,$$
$$[(a+a^\dagger), A(\alpha)] = A(\alpha)(\alpha+\alpha^*), \tag{17}$$

in Eq. (16), we obtain

$$\langle\sin\varphi\rangle = \left\langle 0,0\left|\frac{b+b^\dagger+2(\mathrm{Re}\beta)}{\{[a+a^\dagger+2(\mathrm{Re}\alpha)]^2+[b+b^\dagger+2(\mathrm{Re}\beta)]^2\}^{1/2}}\right|0,0\right\rangle. \tag{18}$$

Upon transforming Eq. (18) to the Schrödinger wave picture and using the oscillator ground-state wave functions,[9] one finds that

$$\langle\sin\varphi\rangle = \frac{1}{\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,\exp[-(x-\mathcal{A})^2-(y-\mathcal{B})^2]\,y/(x^2+y^2)^{1/2}, \tag{19}$$

$$\mathcal{A} \equiv \sqrt{2}(\mathrm{Re}\alpha),\quad \mathcal{B} \equiv \sqrt{2}(\mathrm{Re}\beta).$$

Equations (17)-(19) exhibit the property of the operator $A(\alpha)$ of translating the position of an oscillator. $\langle\sin^2\varphi\rangle$, $\langle\cos\varphi\rangle$, and $\langle\cos^2\varphi\rangle$ are of the same form as (19) with the $\sin\varphi$ in the integrand replaced by the respective operators. The expressions (19) yield the useful knowledge

$$\langle\sin\varphi(\mathcal{A},\mathcal{B})\rangle=\langle\cos\varphi(\mathcal{B},\mathcal{A})\rangle,$$
$$\langle\sin^2\varphi(\mathcal{A},\mathcal{B})\rangle=\langle\cos^2\varphi(\mathcal{B},\mathcal{A})\rangle. \qquad (20)$$

This means that the uncertainty relations (7) and (8) are the same with $\mathcal{A}\leftrightarrow\mathcal{B}$, so only (7) need be studied.

Since the trigonometric operators involve only the real parts of α and β, Eq. (15) tells us that the lowest uncertainty product will be for real α and β. Therefore, we will consider only those states, meaning that we can define ϵ $(0\leqslant\epsilon\leqslant 1)$ such that

$$N=N_x+N_y=\tfrac{1}{2}\mathcal{A}^2+\tfrac{1}{2}\mathcal{B}^2$$
$$\equiv\epsilon N+(1-\epsilon)N, \qquad (21)$$
$$\epsilon=\mathcal{A}^2/(\mathcal{A}^2+\mathcal{B}^2).$$

Using the variables N and ϵ, Eqs. (20) are now of the form

$$\langle\sin\varphi(N,\epsilon)\rangle=\langle\cos\varphi(N,1-\epsilon)\rangle. \qquad (22)$$

Changing (19) to polar coordinates allows an r integration, leaving a φ integration with an error function in the integrand. By a series of tricks these can be further evaluated to yield

$$\left\langle\begin{Bmatrix}\sin\varphi\\ \cos\varphi\end{Bmatrix}\right\rangle=\begin{Bmatrix}\mathcal{B}\\ \mathcal{A}\end{Bmatrix}\frac{2}{\sqrt{\pi}}\int_0^{\pi/2}\cos^2\varphi\exp[-R^2\sin^2\varphi]d\varphi,$$

$$\left\langle\begin{Bmatrix}\sin^2\varphi\\ \cos^2\varphi\end{Bmatrix}\right\rangle=\tfrac{1}{2}e^{-R^2}+\begin{Bmatrix}\sin^2\delta\\ \cos^2\delta\end{Bmatrix}(1-e^{-R^2})$$
$$+\{\pm\}\tfrac{1}{2}(\cos^2\delta-\sin^2\delta)\left(\frac{1-e^{-R^2}}{R^2}-e^{-R^2}\right),$$

$$R^2\equiv\mathcal{A}^2+\mathcal{B}^2=2N,\quad\cos\delta\equiv\mathcal{A}(\mathcal{A}^2+\mathcal{B}^2)^{-1/2}. \qquad (23)$$

From (7), we now define

$$S(N,\epsilon)\equiv\frac{(\Delta L_z)^2(\Delta\sin\varphi)^2}{\langle\cos\varphi\rangle^2}\geqslant\frac{1}{4}. \qquad (24)$$

$S(N,\epsilon)$ was numerically calculated and is plotted in Fig. 1 as a function of N for various values of ϵ. The results agree with the limits for

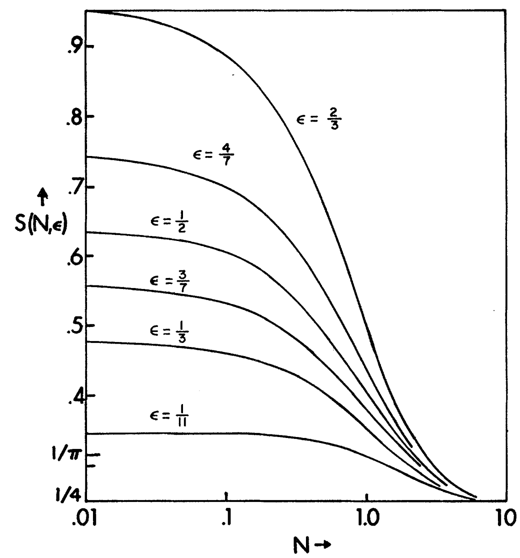

FIG. 1. The uncertainty product $S(N,\epsilon)=(\Delta L_z)^2\times(\Delta\sin\varphi)^2/\langle\cos\varphi\rangle^2$ is shown as a function of N for various values of the parameter ϵ defined in Eq. (21). $S(N,\frac{1}{2})$ is also the uncertainty product $U(N)$ defined in Eq. (26). All expectation values are for the two-dimensional coherent states discussed in the text.

large and small N, which are

$$\lim_{N\to 0}S(N,\epsilon)=\frac{1}{\pi(1-\epsilon)}\geqslant\frac{1}{\pi}\geqslant\frac{1}{4}, \qquad (25)$$
$$\lim_{N\to\infty}S(N,\epsilon)=\tfrac{1}{4}.$$

An uncertainty relation symmetric in sine and cosine, obtained by adding (7) and (8), is

$$U=\frac{(\Delta L_z)^2[(\Delta\sin\varphi)^2+(\Delta\cos\varphi)^2]}{\langle\cos\varphi\rangle^2+\langle\sin\varphi\rangle^2}\geqslant\frac{1}{4}. \qquad (26)$$

It can be shown that U is independent of ϵ and, in fact, is given by[10]

$$U(N)=S(N,\tfrac{1}{2}). \qquad (27)$$

This is not surprising since $\epsilon=\frac{1}{2}$ is a symmetric excitation in the x-y plane.

Our results show that the coherent states do indeed give a low uncertainty product for all N, and a minimum for large N. In a real system we would expect values of ϵ near $\frac{1}{2}$, rather than almost 0 to 1, on physical and statistical grounds.

It is interesting to note the similarity between the coherent-state results for this three-dimensional momentum-angle system, and the one-dimensional coherent-state results for the number-phase system reported by CN. The angular momentum here corresponds to the number operator in CN. The resemblance of the numerical results lends further intuitive understanding to the concept of the S and C operators as being the sine and cosine of the phase angle.

A more detailed account of the results given here and in CN, along with the results of work in progress, will be reported elsewhere.

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NUCLEON FORM FACTORS AND UNIVERSAL VECTOR COUPLING*

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Assuming various combinations of nucleon (N), $\Delta(1236)$ (N^*), and ρ as dominant participants in the low-energy isovector $\pi+\pi\rightarrow N+N$ amplitude, we have calculated the nucleon isovector form factors for moderate momentum transfer. The aim of the calculation was (1) to determine the effect on the theoretical parameters, and on the fit to the data, of the use of the N and N^* and a finite-width ρ, and (2) to produce a form-factor-predicted ρNN vector coupling constant for another test of the hypothesis of universal vector coupling.

Nucleon form factors. – Our initial calculations were made with the usual once-subtracted dispersion relations,[1] using the contributions shown in Fig. 1 for the spectral functions. As can be seen, the pion form factor was assumed to be dominated[2] by the (finite width) rho. It was quickly found that among the possible sets of isovector form factors, F_1^v and F_2^v had rather small subtraction constants while G_E^v and G_M^v did not. Since subtraction constants are undesirable for this process on both theoretical[3] and phenomenological[4] grounds, all subsequent analyses, including those reported here, were made with dispersion relations assumed for the vector current (Dirac) and tensor current (anomalous) form factors F_1^v, F_2^v.

The ρNN vector and tensor coupling constants[2] f_v, f_t were determined by a least-squares fit to a collection[5] of the latest G_E^v, G_M^v data in the range $0 \leq q^2 \leq 22$ F^{-2}, shown in Fig. 2. The two subtraction constants were fixed by the isovector charge and anomalous moment values $F_{1,2}(0)$. Initial calculations also gave a best-fit chi squared[6] which was only one-fifth of its expected value. Since there were 40 data, which were from several different laboratories, it was hardly likely that this incredibly small value was due to either chance or gross overestimation of experimental error. The neutron electric form factor is experimentally consistent with zero for all momentum

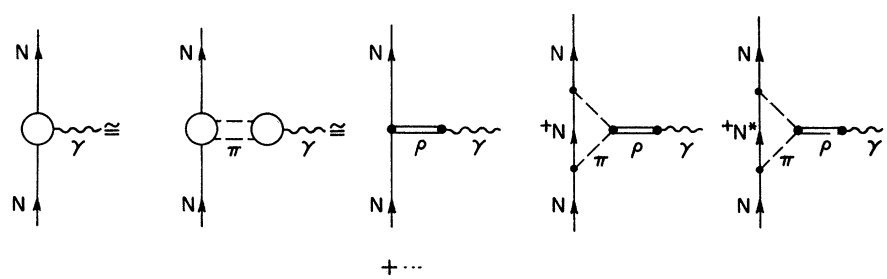

FIG. 1. Two-pion intermediate states in the nucleon isovector form factors, assuming ρ dominance of the pion form factor.