*Research supported in part by the National Science Foundation.

[1]D. G. Sutherland, Phys. Letters 23, 384 (1966); C. Itzykson, M. Jacob, and G. Mahoux, to be published; see also S. Adler, Phys. Rev. Letters 18, 519 (1967).

[2]We use the definition of Adler, Ref. 1: $M^2 \propto (1+2ay)$, where $y=(T-\bar{T})/\bar{T}$, $T$ being the kinetic energy of the odd pion, $\bar{T}$ its mean value.

[3]The commutation relations (1a) and (1b) are extracted, for example, from the $\sigma$ model of J. Schwinger, Ann. Phys. (N.Y.) 2, 407 (1957); M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960). It follows from the Jacobi identity that Eqs. (1a) and (1b) are equivalent, barring any convergence difficulty that may arise in the equal-time limit [F. Buccella, G. Veneziano, R. Gatto, and S. Okubo, Phys. Rev. 149, 1268 (1966)]. Accordingly $\sigma$ cannot vanish since this would imply that the axial-vector current is conserved. L. S. Kisslinger [Phys. Rev. Letters 18, 861 (1967)] has emphasized, quite independently, the importance of the $\sigma$ term.

[4]The conservation of $G$ parity implies that only the $\Delta T=1$ part of the second-order electromagnetic effective Hamiltonian contributes to the matrix element in Eq. (2).

[5]Sutherland's observation was that $C(q_2,q_3)=\frac{1}{2}(A_2-A_3)=0$, as $q_1\to 0$ and $Z_2=Z_3=\mu^2$. Sutherland's contention would have been valid if $B(q_2,q_3)\equiv 0$, or $c_3=0$.

[6]Columbia-Berkeley-Purdue-Wisconsin-Yale Collaboration, Phys. Rev. 149, 1044 (1966).

[7]The quantity in the square bracket on the right-hand side is the $3\pi$ phase space. There is a small correction due to the fact that the Dalitz plot is not a circle (relativistic effect).

[8]The estimate of $\Gamma(\eta\to 2\gamma)$ from the Primakoff effect [C. Bemporad et al., as quoted in D. G. Sutherland, CERN Report No. TH-761] seems to indicate a much larger value. In the present estimate we assume that $\eta$ is an octet member. The effect of the $\eta$–$X^0$ mixing is disucssed by M. Veltman and J. Yellin, Phys. Rev. 154, 1469 (1967).

[9]A. H. Rosenfeld et al., Rev. Mod. Phys. 39, 1 (1967).

[10]G. Feinberg and A. Pais, Phys. Rev. Letters 9, 45 (1962); M. Veltman and J. Yellin, Phys. Rev. 154, 1469 (1967); Adler, Ref. 1.

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# SUM RULE FOR HIGH-ENERGY ELECTRON-PROTON SCATTERING*

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(Received 17 May 1967)

We show that the nonrelativistic quark model leads to an exceptionally simple sum rule involving a form factor determined by high-energy inelastic electron-proton scattering. The simplicity of this result appears to be a unique consequence of the peculiar quark charges.

In spite of its breathtaking crudity the naive quark model enjoys some measure of success.[1] Detectable predictions that really characterize the model's remarkable basic premises are therefore desirable. Here we shall construct a sum rule from the electron-proton cross section that depends critically on the particular fractional charges usually ascribed to quarks. The argument relies on the nonrelativistic nature of the model, through not on any of its finer details. Our prediction Eq. (8) is so strikingly simple that it is hard to resist the conjecture that it can also be derived from a more sophisticated theory of the strong interactions.

Our result stems from the astonishing fact that the naive quark model implies the complete absence of charge and current correlations in the proton. For example, the charge fluctuations are simply

$$\langle\rho(\vec{x})\rho(0)\rangle-\langle\rho(0)\rangle^2=\delta(\vec{x})(2\pi)^{-3}-\langle\rho(0)\rangle^2. \qquad (1)$$

Here $\rho(\vec{x})=\sum_i e_i\sigma(\vec{r}_i-\vec{x})$ is the charge density and $\langle\cdots\rangle$ the spin-averaged expectation value in the proton's ground state, while $e_i$ and $\vec{r}_i$ are the charge and coordinate of the $i$th quark.[2] Equation (1) is actually a special case of the following lemma: Let $A=\sum e_i A_i$, where $A_i$ is an arbitrary operator pertaining solely to the $i$th quark, but independent of its charge, and $B$ another such operator, then

$$\langle AB\rangle=\langle A_1B_1\rangle. \qquad (2)$$

To prove (2) we note that the hadronic Hamiltonian is SU(2) invariant and therefore unchanged by a permutation of any pair of $S=0$ quarks. Because the proton is nondegenerate, its state vector only acquires a phase under such a permutation, whence $\langle A_iB_i\rangle=\langle A_1B_1\rangle$ for all $i$, and $\langle A_iB_j\rangle=\langle A_1B_2\rangle$ for all $i\neq j$. Thus

$$\langle AB\rangle=\langle A_1B_1\rangle\sum_i e_i^2+\langle A_1B_2\rangle\sum_{i\neq j}e_ie_j; \qquad (3)$$

for the proton the first sum in (3) is one, but the second sum vanishes! Q.E.D.[3] Equation (1) now follows from (2) because we use the continuum normalization $\langle\vec{p}|\vec{p}'\rangle=\delta(\vec{p}-\vec{p}')$ in which case $(2\pi)^3\langle\delta(\vec{r}_1)\rangle=1$.

It is well known that fluctuations such as (1) determine the absorption of light by a system.[4] In our problem the electron acts as the light source. Unfortunately, relativistic electron scattering is beset by nontrivial kinematic complications. We therefore begin with a purely pedagogic excercise so as not to hide the essential point. Consider the scattering of a slow and very massive lepton by a proton. If we only take the Coulomb interaction into account, the cross section in the laboratory frame ($L$ frame) for all processes wherein the lepton suffers a three-momentum transfer $\Delta$ and energy loss $\omega$ is

$$\frac{d^2\sigma}{d\Delta^2d\omega}=\frac{(2\pi)^7\alpha^2m_0}{\Delta^4E}\times\tfrac{1}{2}\sum_{n\lambda}\delta(\omega+m-E_n)\delta(\vec{\Delta}-\vec{P}_n)|\langle n|\rho(0)|\lambda\rangle|^2. \qquad (4)$$

Here $m_0$ and $E$ are the mass and incident energy of the lepton, $m$ the mass of the proton, $|\lambda\rangle$ a proton rest state with helicity $\lambda$, $|n\rangle$ an arbitrary state with $Q=B=Y=1$, and $M_n$, $\vec{P}_n$ and $E_n$ the mass, momentum and energy of $|n\rangle$, respectively. We wish to evaluate $d\sigma/d\Delta^2$ and must therefore integrate (4) over $\omega$ with $\Delta$ kept fixed. We now make two further assumptions (known as the closure approximation): (i) $E\gg\Delta^2/2m_0$, and (ii) $\langle n|\rho|\lambda\rangle$ is negligible for highly excited states, i.e., if $|M_n-m|>\Delta$. This last assumption is certainly not correct for a real proton. Nevertheless, once it is made we can extend the integral to $\omega=\infty$ and use the completeness of the set $|n\rangle$ to obtain

$$\frac{d\sigma}{d\Delta^2}=\frac{(2\pi)^4\alpha^2m_0}{\Delta^4E}\int d^3x\,e^{-i\vec{\Delta}\cdot\vec{x}}\langle\rho(\vec{x})\rho(0)\rangle.$$

Recalling (1), we have $d\sigma/d\Delta^2=2\pi\alpha^2m_0/E\Delta^4$, which is just Rutherford's formula for scattering by a point charge. This means that the decrease of the elastic cross section due to the elastic form factor is exactly compensated by inelastic scattering.

In high-energy scattering the electron's energy loss must be less than $\Delta$, and the excitation energies are comparable with the proton's mass. The closure approximation is therefore hopeless at fixed $\Delta$, and instead one must work at constant $(k_\mu-k_\mu')^2=q^2$, where $k_\mu$ and $k_\mu'$ are the initial and final electron four-momenta.[5] Moreover, closure cannot be applied unambiguously to the cross section itself. The correct procedure leads to sum rules for form factors, and we must therefore record the definitions of these quantities.[6] In the $L$ frame the cross section is

$$\frac{d^3\sigma}{d\Omega dk'}=\frac{4\alpha^2}{q^4}k'^2\cos^2\frac{\theta}{2}\left[W_2(q^2,\nu)+2W_1(q^2,\nu)\tan^2\frac{\theta}{2}\right],$$

where $\theta$ is the scattering angle, $q^2=-4kk'\sin^2\frac{1}{2}\theta$, and $\nu=p\cdot q/m$, $p_\mu$ being the target's momentum; $\nu$ is the energy loss in $L$. Let $e_\mu$ and $u_\mu$ be space-like unit vectors; furthermore, let $e\cdot u=e\cdot q=e\cdot p=u\cdot p=0$, and $u\cdot q=Q$, where $Q^2=-q^2$. The longitudinal and transverse portions of the hadronic current $J_\mu(x)$ are then $J_L=u\cdot J$ and $J_T=e\cdot J$, respectively, and the corresponding invariant form factors are

$$W_1(q^2,\nu)=\gamma(2\pi)^2\int d^4x\,e^{iq\cdot x}\langle J_T(x)J_T(0)\rangle_p, \qquad (5)$$

$$-(\nu^2/q^2)W_2(q^2,\nu)=\gamma(2\pi)^2\int d^4x\,e^{iq\cdot x}\langle J_L(x)J_L(0)\rangle_p. \qquad (6)$$

Here $\gamma=p_0/m$, and the subscript $p$ indicates that the expectation value is in a one-proton state with momentum $p_\mu$.

Our aim is to integrate $W_2$ over $\nu$ at fixed $q^2$. In the $L$ frame it is very difficult to do this, because the energy and momentum difference between the proton state and $|n\rangle$ both depend on $q^2$ and $\nu$. As Dashen and Gell-Mann have stressed,[7] the kinematics is greatly simplified if one transforms to a frame in which the proton has a very large momentum.[8] To be precise, let $\bar{M}$ be the largest mass that contributes significantly to $\int d\nu\,W_2$ at the specified value of $q^2$, and $\bar{\nu}$ the corresponding $\nu$, i.e., $\bar{\nu}=(\bar{M}^2-q^2-m^2)/2m$; then we require $\gamma\gg\bar{\nu}/Q$, and we shall call a frame of this type an $F$ frame.

In $F$ we are free to choose $q$, $p$, and $u$ to suit our needs, provided of course that $q^2$ and $q\cdot p$ have the desired values. A convenient choice is $p^\mu=(\gamma m,0,0,p_z)$, $q^\mu=(\nu/\gamma,\vec{Q})$ with $\vec{Q}$ in the $x$-$y$ plane, $|\vec{Q}|^2=-q^2$, and $u^\mu=(0,-\hat{Q})$. If $|p\rangle$ is a one-proton state and $\langle n|J|p\rangle_\infty$ some matrix element in an $F$ frame, then[7] to lowest order in $1/\gamma$ the energy difference between $|p\rangle$ and $|n\rangle$ is $(M_n^2-m^2-q^2)/2m\gamma\equiv\nu_n/\gamma$, and the square of the three-momentum transfer equals $-q^2$. Taking advantage of these facts, and of our definitions, we can write the right-side of (6) as

$$\gamma^2(2\pi)^3\tfrac{1}{2}\sum_{\lambda n}\delta(\nu-\nu_n)$$
$$\times\langle p\lambda|\int d^3x\,e^{-i\vec{Q}\cdot\vec{x}}\hat{Q}\cdot\vec{J}(\vec{x})|n\rangle_\infty\cdot\langle n|\hat{Q}\cdot\vec{J}(0)|p\lambda\rangle_\infty.$$

These matrix elements can be simplified because current conservation implies $\langle p|\hat{Q}\cdot\vec{J}|n\rangle_\infty=(\nu_n/Q\gamma)\langle p|\rho|n\rangle_\infty$. Keeping $\vec{Q}$ constant fixes $q^2$ in $F$, and we can now integrate freely over $\nu$. Our final result is then

$$\int_0^\infty d\nu\,W_2(q^2,\nu)=(2\pi)^3\int d^3x\,e^{-i\vec{Q}\cdot\vec{x}}\langle\rho(\vec{x})\rho(0)\rangle_\infty. \quad (7)$$

In spite of its appearance this expression is Lorentz invariant.

This seems to be as far as logic can take us. If the integrand in (7) contained a current commutator, current algebra would allow an unambiguous, Lorentz-covariant determination of the expectation value. But if we transform an equal-time product from $F$ to another frame, it looses its equal-time character, and detailed dynamical information is then required for the evaluation of the expectation value. We are therefore forced to throw caution aside: We assume that we can evaluate the right side of (7) with the nonrelativistic quark model. Once this is accepted we can transform to any frame (e.g., to $L$) and retain the equal-time property of the operator product. Equation (7) is then evaluated by using (1) once more. After separating out the elastic contribution we obtain

$$\int_{\nu_0}^\infty d\nu\,W_2(q^2,\nu)=1-\frac{G_E^2(q^2)-(q^2/4m^2)G_M^2(q^2)}{1-q^2/4m^2}, \quad (8)$$

$$\simeq 1-\frac{1+2\cdot 2t}{(1-0.29t)(1+1.4t)^4}, \quad (9)$$

where $\nu_0$ is the inelastic threshold. In writing (9) we have used the empirical fits[9] $G_E=G_M/\mu_p=(1+1.4t)^{-2}$, where $t=-q^2\,(\mathrm{BeV}/c)^{-2}$. Equation (8) is our main result.[10]

A photoproduction sum rule follows from (8) because[11]

$$-\frac{\partial W_2(q^2,\nu)}{\partial q^2}\bigg|_{q^2=0}=\frac{1}{(2\pi)^2\alpha}\frac{\sigma(\nu)}{\nu}, \quad (10)$$

where $\sigma(\nu)$ is the total photoproduction cross section. Thus (8) also implies

$$\int_0^\infty\sigma(\nu)\frac{d\nu}{\nu}=\frac{\pi^2\alpha}{m^2}\left[\frac{4}{3}m^2\langle r^2\rangle+1-\mu_p^{\ 2}\right]=0.42\text{ mb}. \quad (11)$$

Evaluation[12] of the integral in (11) with the data below $\nu=5$ BeV gives 0.65 mb. The agreement between theory and experiment is therefore only qualitative.

How is one to assess our sum rule? Two extreme points of view are: (I) The naive quark model is idiotic, and whatever success it has had is accidental—hence (8) should be ignored. (II) The model has some overlap with the truth, and when it is used judiciously in the Fubini-Furlan frame, this overlap is actually quite appreciable; hence one should look for a derivation of (8) that a well-educated person could believe. Between these extremes a spectrum of possible attitudes suggest itself. Thus Bjorken[13] has pointed out that the naive model cannot account for diffraction production of $\rho^0$, which is the dominant electrodynamic process at high energy[14]; in fact, the sum rule converges very slowly at best because of this process. One could also be more conservative, and say that the naive model has too few degrees of freedom to account for anything beyond the production of nucleon isobars. If one is of this opinion, one would isolate these resonances from the inelastic cross section to see whether they themselves satisfy the sum rule. The sign and magnitude of the discrepancy between (11) and the present data indicate that this may be the most reasonable position [aside from (I), perhaps]. Hopefully, forthcoming experiments on electroproduction will settle the issue.

I am greatly indebted to L. N. Hand; his provocative questions led to this investigation. I have also received very useful suggestions from K. Berkelman, P. A. Carruthers, H. Goldbert, and D. R. Yennie. I should like to thank J. D. Bjorken for communicating his own results prior to publication, and for his enlight-

ening comments on an earlier version of this paper.

*Work supported in part by the National Science Foundation.

[1]For reviews of the model, see the reports by R. H. Dalitz and L. Van Hove, in Proceedings of the Thirteenth International Conference on High Energy Physics, Berkeley, California, 1966 (University of California Press, Berkeley, 1967).

[2]We assume that the quark form factors do not depart from unity.

[3]This argument shows that (2) generalizes to $\langle A(t)B\rangle = \langle A_1(t)B_1\rangle$, where $A(t)$ is $A$ in the Heisenberg picture; furthermore, it applies separately to each proton spin state. The cancellation of all pair correlations depends on the SU(3) charge assignments, even though SU(3) symmetry breaking does not effect our argument. Of course naive SU(6) will also lead to (2), but as we have shown, (2) still holds if the interactions are spin dependent provided only that they are charge independent. Also note that for the neutron, $\langle AB\rangle = \frac{2}{3}[\langle A_1B_1\rangle - \langle A_1B_2\rangle]$, and the neutron sum rule therefore involves an unknown pair-correlation function.

[4]Cf., e.g., L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media (Pergamon Press, London, 1960), Chap. 14.

[5]J. D. Bjorken, Phys. Rev. Letters 16, 408 (1966), has already derived a sum rule inequality for the sum of the neutron and proton cross sections under the same kinematic conditions. His work is based on SU(2) current algebra, and he only obtains an inequality because the effect of the isoscalar current is not included in this approach. Our result does not appear to have any simple relationship with current algebra; on the other hand, the quark model specifies the isoscalar current. In Bjorken's work, and also in the intimately related paper by S. L. Adler, Phys. Rev. 143, 1144 (1966), unsubtracted dispersion relations are an essential tool. It would seem that the closure approximation is equivalent to the assumption of unsubtracted dispersion relations. C. G. Callen informs me that he has also come to a similar conclusion some time ago.

[6]The literature can be traced from J. D. Bjorken and J. D. Walecka, Ann. Phys. (N.Y.) 38, 35 (1966). Note that these authors call $W_i$ what we here call $mW_i$.

[7]R. F. Dashen and M. Gell-Mann, in Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1966 (W. H. Freeman & Company, San Francisco, California, 1966); Phys. Rev. Letters 17, 340 (1966).

[8]S. Fubini and G. Furlan, Physics 1, 229 (1965).

[9]J. R. Dunning, K. W. Chen, A. A. Cone, G. Hartwig, N. F. Ramsey, J. K. Walker, and R. Wilson, Phys. Rev. 141, 1286 (1966).

[10]One might think that there should also be a similar sum rule for $W_1$. This is not so for at least two reasons: (1) $W_1(0,\nu)$ is proportional to $\nu\sigma(\nu)$, and $\int W_1(0,\nu)d\nu$ is therefore guaranteed to diverge; (2) the expression corresponding to (7) for $W_1$ is proportional to $\gamma^2$, and the transition to the nonrelativistic model is therefore dangerous at best. Nevertheless, it is still true that Eq. (2) implies the absence of correlations from $\langle J_T(\vec{x})J_T(0)\rangle$, because the naive model asserts that the current comes purely from quark charge convection and the individual magnetic moments, in which case $J_T$ is a linear form in the quark charges. We have not yet found any sensible consequences of this absence of transverse current correlations.

[11]I am very grateful to D. R. Yennie for bringing this identity to my attention. It is proven in S. D. Drell and J. D. Walecka, Ann. Phys. (N.Y.) 28, 18 (1964).

[12]For $\nu < 0.4$ BeV we have used the compilation by J. T. Beale, S. D. Ecklund, and R. L. Walker, California Institute of Technology Synchrotron Report No. 42, 1966 (unpublished), especially p. 107. For $0.4 < \nu < 5.0$ BeV we have used Fig. 11 in Cambridge Bubble Chamber Group, Phys. Rev. 155, 1477 (1967). The various energy regimes give the following contribution to the integral: $\nu < 0.4$ BeV, 0.28 mb; $0.4 < \nu < 1$ BeV, 0.21 mb; and $1 < \nu < 5$ BeV, 0.16 mb. If one follows Bjorken's suggestion (private communication), and subtracts the $\rho^0$-production cross section [Cambridge Bubble Chamber Group, Phys. Rev. 146, 994 (1966)], the integral is reduced by about 0.1 mb. The effect of subtracting the $\rho^0$ cross section is therefore not dramatic if one stops integrating at $\nu = 5$ BeV, but there is no question that the convergence of the integral would be in serious doubt without this subtraction.

[13]Bjorken, Ref. 12.