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η DECAY AND CURRENT ALGEBRA

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It has been argued[1] that in the usual picture of current-current electromagnetic interaction, current algebra and the assumption of a linear dependence of the matrix element on the energy variables imply that the decays $\eta \to 3\pi$ are forbidden. In this note we show that, when the partially conserved axial-vector current (PCAC) hypothesis and current algebra are systematically and carefully applied, one finds, within the framework of conventional electrodynamics, that (1) the decays $\eta \to 3\pi$ are allowed; (2) the slope[2] a of the Dalitz plot for $\eta \to \pi^+ + \pi^- + \pi^0$ is -0.49, in excellent agreement with experiment; and (3) the rate for $\eta \to 3\pi^0$ is at most of order of 10^2 eV. In addition to the usual assumptions about PCAC and the current algebra, one essential assumption in our calculation is that[3]

$$[Q_5^{\alpha}(x_0), D^{\beta}(x)] = i\delta^{\alpha\beta}\sigma(x), \qquad (1a)$$

$$[Q_5^{\alpha}(x_0), \sigma(x)] = -iD^{\alpha}(x), \qquad (1b)$$

where $Q_5^{\alpha}(x_0) = \int d^4x\, A_0^{\alpha}(x, x_0)$ is the axial charge, $D^{\alpha}(x) \equiv \partial^{\mu} A_{\mu}^{\alpha}(x)$, and $\sigma(x)$ is an isoscalar, scalar field. Our final results make no reference to the detailed properties of $\sigma(x)$, however.

In this picture the branching ratio $\Gamma(\eta \to 3\pi^0)/\Gamma(\eta \to \pi^+ + \pi^- + \pi^0)$ is of course predicted to be approximately $\frac{3}{2}$. We shall further comment on this point later.

To exploit PCAC and current algebra systematically, we define a symmetric off-shell matrix element for $\eta(p) \to \pi^{\alpha}(q_1) + \pi^{\beta}(q_2) + \pi^{\gamma}(q_3)$:

$$T^{\alpha\beta\gamma}(q_1, q_2, q_3) = (2\pi)^{\frac{3}{2}}(2M\eta)^{\frac{1}{2}}\left[\prod_{i=1}^{3} (q_i^2 - \mu^2)\int d^4x_i \exp(iq_i x_i)\right]\langle 0 | T\{D^{\alpha}(x_1) D^{\beta}(x_2) D^{\gamma}(x_3) H_{\rm em}(0)\} | \eta\rangle, \qquad (2)$$

where[4]

$$H_{\rm em}(0) = e^2 \int dx\, D_{\mu\nu}(x) T\{V_{\mu}^{3}(x) V_{\nu}^{0}(0)\},$$

V_μ^3, V_ν^0 being the isovector and isoscalar components of the electromagnetic current. We are interested in varying the pion momenta, q_i, from zero (where current algebra makes definite statements about the off-shell matrix element) to physical values where $q_i^2=\mu^2$. There are two alternative ways of carrying out this program within the "linear" matrix-element approximation:

(1) Four-momentum conservation $p=\sum q_i$ is not imposed on the matrix element in the extrapolation, and the matrix element has poles in $(\sum q_i)^2$ and q_i^2. The pole terms are removed from the amplitude and the remainder is approximated by a linear expansion in all variables. The coefficients in this expansion are determined in the region $q_1, q_2, q_3 \simeq 0$ by current algebra and PCAC.

(2) The q_i are varied with the constraint $p=\sum q_i$. In this case the pion pole $[(\sum q_i)^2-\mu^2]^{-1}$ takes a constant value, while η pole $(q_i^2-M_\eta^2)^{-1}$ is slowly varying in the region $0\leqslant q_i^2\leqslant\mu^2$. We may therefore assume that the entire matrix element may be approximated by a linear expansion.

Both methods have been used and yield essentially identical results, demonstrating that the linear approximation is internally consistent. For brevity, we will outline the second method. Both techniques will be described fully elsewhere.

We write

$$T^{\alpha\beta\gamma}(q_1,q_2,q_3)=\delta_{3\alpha}\delta_{\beta\gamma}A_1(s_i,Z_i)+\delta_{3\beta}\delta_{\gamma\alpha}A_2(s_i,Z_i)+\delta_{3\gamma}\delta_{\alpha\beta}A_3(s_i,Z_i), \tag{3}$$

where A_1, A_2, and A_3 are related to each other by a cyclic permutation of the variables $s_i=(p-q_i)^2$, $Z_i=q_i^2$, which satisfy the constraint $\sum s_i=M_\eta^2+\sum Z_i$. With the assumption of linear dependence of A_i on the variables s_j and Z_j and using the explicit Bose symmetry of the off-shell amplitude (2), we have

$$A_1=c_0+c_1(s_2+s_3)+c_2s_1+c_3(Z_2+Z_3). \tag{4}$$

We take the $q_1\to0$ limit of Eq. (2) and use Eq. (1a) to obtain

$$T^{\alpha\beta\gamma}(0,q_2,q_3)=i\mu^2(Z_2-\mu^2)(Z_3-\mu^2)\{\delta_{3\gamma}\delta_{\alpha\beta}[B(q_2,q_3)+C(q_2,q_3)]+\delta_{3\beta}\delta_{\gamma\alpha}[B(q_3,q_2)+C(q_3,q_2)]\}, \tag{5}$$

where

$$B(q_2,q_3)=(2\pi)^{\frac{3}{2}}(2M_\eta)^{\frac{1}{2}}\int d^4x d^4y\, e^{i(q_2x+q_3y)}\langle 0|T\{\sigma(x)D^0(y)H_{\rm em}(0)\}|\eta\rangle, \tag{6}$$

and

$$C(q_2,q_3)=(2\pi)^{\frac{3}{2}}(2M_\eta)^{\frac{1}{2}}\int d^4x d^4y\, e^{i(q_2x+q_3y)}\tfrac{1}{2}\{\langle 0|T[D^+(x)D^-(y)H_{\rm em}(0)]|\eta\rangle-(x\leftrightarrow y)\}. \tag{7}$$

Comparing Eqs. (3) and (5), we find that

$$A_1=0,\quad \text{as } q_1\to0. \tag{8}$$

Furthermore, the structure of $B(q_2,q_3)$ and $C(q_2,q_3)$ requires that[5]

$$A_2=A_3=0,\quad \text{as } q_1\to0 \text{ and } Z_2, Z_3\to\mu^2. \tag{9}$$

The conditions (8) and (9) give

$$A_i=c_1[M_\eta^2-s_i+(1-M_\eta^2/\mu^2)Z_i]. \tag{10}$$

On the mass shell, we have

$$A_i(s_i)=-c_1M_\eta^2(1-2w_i/M_\eta), \tag{11}$$

where w_i is the total energy of pion i in the η rest system. This gives the slope[2] of the Dalitz plot as

$$a=-2(M_\eta-3\mu)/M_\eta\simeq-0.49$$

which is to be compared with the experimental value[6]

$$a=-0.478\pm0.038.$$

To find the strength of the amplitude we must determine the parameter c_1. From Eqs. (5) and (10), we find

$$\lim_{\substack{q_1\to 0\\ Z_2\to\mu^2}}(A_2+A_3)=i(2\pi)^{\frac{3}{2}}(2M_\eta)^{\frac{1}{2}}\mu^2\lim_{Z_2\to\mu^2}(Z_2-\mu^2)\int d^4x d^4y\, e^{i(q_3x+q_2y)}$$

$$\times[\langle 0|T\{\sigma(x)D^0(y)H_{\rm em}(0)\}|\eta\rangle+(x\leftrightarrow y)],$$

or

$$-(M_\eta/\mu)^2c_1=i(2\pi)^{\frac{3}{2}}(2M_\eta)^{\frac{1}{2}}\mu^2\lim_{Z_2\to\mu^2}(Z_2-\mu^2)\int d^4xd^4y\, e^{i(q_3x+q_2y)}\langle 0|T\{\sigma(x)D^0(y)H_{\rm em}(0)\}|\eta\rangle. \quad (12)$$

We estimate the expression on the right of Eq. (12) using PCAC (i.e., instead of the limit $Z_2\to\mu^2$, we take the limit $q_2\to 0$) and obtain, with the help of Eq. (1b),

$$M_\eta^2c_1=\mu^6\int d^4x\, e^{ipx}\langle 0|T\{D^0(x)H_{\rm em}(0)\}|\eta\rangle(2\pi)^{\frac{3}{2}}(2M_\eta)^{\frac{1}{2}}. \quad (13)$$

[In applying PCAC, we insist on $p=\sum q_i$. Therefore, as $q_1\to 0, q_2\to 0$ we must have $q_3\to p$. In order for Eq. (13) to be valid, there should not be a strong singularity in $Z_3=q_3^2$ of the right-hand side of Eq. (12) in the range of $0\le Z_3\le M_\eta^2$, corresponding to a scalar excitation.] The expression (13) may be related to the K^+-K^0 and π^+-π^0 mass differences under SU(3), but the exact relation becomes ambiguous in the presence of SU(3) symmetry breaking. In the pion-pole approximation one obtains

$$M_\eta^2c_1=f_\pi\mu^6M_\eta^2(\mu^2-M_\eta^2)^{-1}(\Delta M_K^2/\sqrt{3})(1-\Delta M_\pi^2/\Delta M_K^2), \quad (14)$$

where $\Delta M_K^2=M_{K^+}^2-M_{K^0}^2$, $\Delta M_\pi^2=M_{\pi^+}^2-M_{\pi^0}^2$, and f_π is the pion decay constant. [Estimates of Eq. (13) may differ from (14) by as much as (μ/M_η), depending on the way one compares $\langle\eta|H_{\rm em}|\pi\rangle$ and $\langle K,\pi|H_{\rm em}|K,\pi\rangle$. Equation (14) is the largest value one can obtain within reason.]

The rate for $\eta\to 3\pi^0$ is given by[7]

$$\Gamma(\eta\to 3\pi^0)\simeq\frac{1}{2M}\left|\left(\frac{1}{\mu^2f_\pi}\right)^3(A_1+A_2+A_3)\right|^2\left[\frac{1}{3!}\frac{1}{(2\pi)^2}\frac{(M_\eta-3\mu)^2}{48\sqrt{3}}\right], \quad (15)$$

and the estimate (14) gives

$$\Gamma(\eta\to 3\pi^0)\simeq 1.6\times 10^2 \text{ eV}.$$

Experimentally the absolute rate for this decay is not known. Indirect information is obtained by estimating the amplitude for $\eta\to 2\gamma$ from that of $\pi^0\to 2\gamma$ by SU(3), and using the experimental branching ratio $\Gamma(\eta\to 3\pi^0)/\Gamma(\eta\to 2\gamma)\simeq\frac{2}{3}$. The rate so obtained[8] is

$$\Gamma(\eta\to 3\pi^0)\simeq\tfrac{2}{3}\cdot\tfrac{1}{3}(M_\eta/\mu)^3\Gamma(\pi^0\to 2\gamma)\simeq 1.1\times 10^2 \text{ eV}.$$

The two independent estimates of $\Gamma(\eta\to 3\pi^0)$ are in reasonable agreement. Since we have assumed the usual second-order electromagnetic interaction, we predict a branching ratio $\Gamma(\eta\to 3\pi^0)/\Gamma(\eta\to\pi^+ +\pi^- +\pi^0)$ of roughly 1.5. The present ratio quoted by Rosenfeld et al.[9] is 0.94 ± 0.16. This discrepancy between the theoretical and experimental values of the branching ratio has prompted several theoretical speculations.[10] There does not seem to be, however, an experimental consensus as to the value of this branching ratio. Experimental clarification of this point is of great interest. If the branching ratio does indeed turn out to be less than, say, 1.2, while there is no appreciable quadratic or higher term present in the Dalitz-plot distribution, one must seriously entertain the possibility of some $\Delta I\ge 3$ interactions in the η decay as emphasized by Feinberg and Pais.[10] Even in this case, we have shown that the conventional electrodynamic effect is not negligible, and this effect does produce the right slope in the Dalitz plot. Thus if the $\Delta I\ge 3$ interaction is such as to produce the correct slope, as suggested by Adler,[10] the inclusion of the effect we have discussed will not destroy the agreement between theory and experiment.

In our approach, we find that the process in which a soft pion is emitted from the (nonexistent) $\eta\to 2\pi$ vertex is forbidden, and that the decay $\eta\to 3\pi$ is dominated by the pion-pole diagram $\eta\to(\pi)\to 3\pi$ and a related remainder term. We shall describe below a "chiral dynamics"[11] model which incorporates the pion-pole contribution and yields precisely our current-algebra result.

A simple phenomenological Lagrangian which satisfies the chiral SU(2)⊗SU(2) algebra, PCAC, and the postulated commutation relations (1a) and (1b) is obtained by writing

$$\mathcal{L}_0=-\tfrac{1}{2}\{\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}+\partial_\mu\sigma\partial^\mu\sigma\}-f_\pi\mu^2\sigma, \tag{16}$$

where $\sigma(x)$ is to be interpreted as a dependent field, given by

$$\sigma(x)=[f_\pi^{\,2}-\vec{\pi}^2(x)]^{1/2}=f_\pi-\frac{\vec{\pi}^2}{2f_\pi}-\frac{(\vec{\pi}^2)^2}{8f_\pi^{\,3}}+\cdots. \tag{17}$$

In this model the vector and axial-vector current are given by

$$V_\mu^{\ \alpha}=-\epsilon^{\alpha\beta\gamma}\pi_\beta\partial_\mu\pi_\gamma,$$

$$A_\mu^{\ \alpha}=\sigma\partial_\mu\pi^\alpha-\pi^\alpha\partial_\mu\sigma. \tag{18}$$

It is not too difficult to verify that the currents in Eq. (18) satisfy all the conditions we have stipulated. To fourth order in the (independent) pion field, we have

$$\mathcal{L}_0=-\tfrac{1}{2}(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}-\mu^2\vec{\pi}^2)-\frac{1}{8f_\pi^{\,2}}[\partial_\mu(\vec{\pi}^2)\partial^\mu(\vec{\pi}^2)-\mu^2(\vec{\pi}^2)^2], \tag{19}$$

$$A_\mu^{\ \alpha}=f_\pi\partial_\mu\pi^\alpha-\frac{1}{2f_\pi}\pi^\alpha\partial_\mu(\vec{\pi})^2-\frac{1}{2f_\pi}\vec{\pi}^2\partial_\mu\pi^\alpha+O(\pi^4). \tag{20}$$

To effect the decay $\eta\to 3\pi$ we construct a phenomenological $\eta\pi$ coupling which has the same transformation properties under the chiral algebra as $H_{\rm em}$. The unique coupling which gives a linear matrix element is (assuming $[Q_5,\eta]=0$)

$$\mathcal{L}_I=gA_\mu^{\ 3}\partial^\mu\eta. \tag{21}$$

Application of the usual Feynman rule to $\mathcal{L}_0+\mathcal{L}_I$ to lowest order in pion fields ($\sim\pi^3$) gives precisely the result of Eq. (14). [An ambiguity in this approach also exists in relating g to ΔM_K^2 and ΔM_π^2.]

We have examined various applications of the current algebra in the light of the importance of the σ term. We find that most results are unchanged, except for the decays $K\to 3\pi$. The result of Hara and Nambu[12] and Abarbanel[13] depended on the omission of the σ terms, usually referred to as "final-state interactions." Careful analysis of the decay $K\to 3\pi$ including the σ terms yields results that are slightly different from those of Refs. 12 and 13. The current-algebra results for the decays $K\to 3\pi$ and application of the chiral SU(3)⊗SU(3) dynamics method to the decays $K\to 3\pi$ will be reported elsewhere, with a detailed description of the present work.

After the completion of this work, we found that a calculation similar to that outlined in Eqs. (1)-(11) has been made by Dolgov, Vainshtein, and Zakhorov. These authors, however, relate the strength of the decay amplitude to the η pole term and the η-π scattering amplitude. We find this unacceptable.

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[4]The conservation of G parity implies that only the $\Delta T=1$ part of the second-order electromagnetic effective Hamiltonian contributes to the matrix element in Eq. (2).

[5]Sutherland's observation was that $C(q_2,q_3)=\frac{1}{2}(A_2-A_3)=0$, as $q_1\to 0$ and $Z_2=Z_3=\mu^2$. Sutherland's contention would have been valid if $B(q_2,q_3)\equiv 0$, or $c_3=0$.

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[7]The quantity in the square bracket on the right-hand side is the 3π phase space. There is a small correction due to the fact that the Dalitz plot is not a circle (relativistic effect).

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SUM RULE FOR HIGH-ENERGY ELECTRON-PROTON SCATTERING*

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We show that the nonrelativistic quark model leads to an exceptionally simple sum rule involving a form factor determined by high-energy inelastic electron-proton scattering. The simplicity of this result appears to be a unique consequence of the peculiar quark charges.

In spite of its breathtaking crudity the naive quark model enjoys some measure of success.[1] Detectable predictions that really characterize the model's remarkable basic premises are therefore desirable. Here we shall construct a sum rule from the electron-proton cross section that depends critically on the particular fractional charges usually ascribed to quarks. The argument relies on the nonrelativistic nature of the model, through not on any of its finer details. Our prediction Eq. (8) is so strikingly simple that it is hard to resist the conjecture that it can also be derived from a more sophisticated theory of the strong interactions.

Our result stems from the astonishing fact that the naive quark model implies the complete absence of charge and current correlations in the proton. For example, the charge fluctuations are simply

$$\langle\rho(\vec{x})\rho(0)\rangle-\langle\rho(0)\rangle^2=\delta(\vec{x})(2\pi)^{-3}-\langle\rho(0)\rangle^2. \qquad (1)$$

Here $\rho(\vec{x})=\sum_i e_i\sigma(\vec{r}_i-\vec{x})$ is the charge density and $\langle\cdots\rangle$ the spin-averaged expectation value in the proton's ground state, while e_i and $\vec{r}_i$ are the charge and coordinate of the ith quark.[2] Equation (1) is actually a special case of the following lemma: Let $A=\sum e_i A_i$, where A_i is an arbitrary operator pertaining solely to the ith quark, but independent of its charge, and B another such operator, then

$$\langle AB\rangle=\langle A_1B_1\rangle. \qquad (2)$$