[6]Under the above experimental conditions, if the distribution function for the transverse velocity assumes the form $\delta(\vec{V}-\vec{V}_\perp)$, the electron's driven gyrofrequency must be ω_x'. Collisions will randomize this distribution on a time scale >1 μsec in this experiment; however, the confinement time of the plasma is itself <10 μsec. From the mode spacing of Fig. 2. we deduce that $\frac{1}{2}mV_\perp^2 \sim 10$ eV which is about four times the unperturbed mean electron thermal energy (determined by a Langmuir probe). Thus the assumption of the cold plasma model is not unreasonable.

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REFLECTION OF A PLASMA WAVE AT AN ELECTRON SHEATH*

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An electrostatic wave propagating into a region of decreasing electron density is generally taken to be absorbed with small reflection,[1] as a consequence of the onset of strong Landau damping. The knowledge of the reflection coefficient, however small, is important in the attempt to suppress convectively unstable waves by limiting the plasma size. For example, there have been a number of calculations of the critical length of a mirror machine subject to a loss-cone instability.[2-5] These calculations have yielded small reflection coefficients. However, they have omitted certain effects of electron reflection at the sheath which we will show below may, for certain density distributions, lead to a coefficient of order one.

As a basis for calculation, we will adopt the model used by Berk, Rosenbluth, and Sudan (BRS).[6] This is a one-dimensional plasma with electron density uniform for $-\infty<x<0$ but going to 0 for $x\to+\infty$, and a plasma wave impinging from the left whose frequency is slightly above ω_p, the value of the plasma frequency for $x<0$. In the region where the plasma wave is only weakly damped, we will assume that the static potential, $-m\Phi(x)/e$, is slowly varying on the scale of the plasma wavelength. Further to the right, where the wave is heavily damped, the potential may have arbitrary variation. In this latter sense, we generalize the model of BRS in which the potential was taken to be slowly varying everywhere.

When all quantities vary only in the x direction, BRS obtain the following equation for the perturbed electric field $\mathcal{E}(x)$:

$$\mathcal{E}(x)=-\frac{\omega_p^2}{i\omega}\int_{\Phi(x)}^{\infty}dE\,\frac{\partial F}{\partial E}\left[\int_{-\infty}^{x}dx'\,\mathcal{E}(x')\exp\left(i\omega\int_{x'}^{x}\frac{dx''}{v(x'')}\right)+\int_{x}^{x_0}dx'\,\mathcal{E}(x')\exp\left(i\omega\int_{x}^{x'}\frac{dx''}{v(x'')}\right)\right.$$
$$\left.-\exp\left(2i\omega\int_{x}^{x_0}\frac{dx''}{v(x'')}\right)\int_{-\infty}^{x_0}dx'\,\mathcal{E}(x')\exp\left(i\omega\int_{x'}^{x}\frac{dx''}{v(x'')}\right)\right] \quad (1)$$

In Eq. (1) we have set $E=\frac{1}{2}v^2+\Phi(x)$ and $v(x)=\sqrt{2}[E-\Phi(x)]^{1/2}$ and defined x_0 to be the turning point, $v(x_0)=0$. $F(E)$ is the Maxwell-Boltzmann distribution. All time-dependent quantities have been taken to vary as $\exp(-i\omega t)$ with $\mathrm{Im}\omega>0$. Whenever appropriate, it is understood that the limit $\mathrm{Im}\omega\to0^+$ is to be taken.

Because of the assumed properties of $\Phi(x)$, we look for solutions of Eq. (1) of the form

$$\mathcal{E}(x)=\mathcal{E}_+(x)\exp(i\int_0^x k\,dx')+\mathcal{E}_-(x)\exp(-i\int_0^x k\,dx'), \quad (2)$$

where

$$\frac{1}{k\mathcal{E}_\pm}\frac{d\mathcal{E}_\pm}{dx}\sim\frac{1}{k^2}\frac{dk}{dx}\sim\frac{1}{k\Phi}\frac{d\Phi}{dx}\equiv\delta\ll1$$

within the region of weak damping. The local wave number $k(x)$ will be taken to be real, and such effects as spatial Landau damping will be included in $\mathcal{E}_\pm(x)$. For weak damping this is equivalent to dealing with complex k, but proves more convenient.

After substituting Eq. (2) into Eq. (1), we integrate by parts twice the first two x' integrals of Eq. (1) in a manner identical to BRS except that we disregard the contribution from the point x_0. [For all electrons except those of very small E, x_0 is well within the sheath and so $\mathcal{E}(x_0)$ is negligibly small.] We then introduce the function

$$\epsilon(\omega,k,x) \equiv 1-\frac{\omega_p^{\,2}}{\omega}\,\mathrm{Re}\int_{\Phi(x)}^{\infty} dE\,\frac{\partial F}{\partial E}\left(\frac{1}{k-\omega/v(x)}-\frac{1}{k+\omega/v(x)}\right), \tag{3}$$

where Re denotes the real part, and here, as below, the limit $\mathrm{Im}\omega \to 0^+$ is to be taken. By defining $k(x)$ to satisfy

$$\epsilon(\omega, \pm k(x), x) = 0,$$

the nominally lowest-order result of the integration by parts cancels out. The next order gives an equation containing both $d\mathcal{E}_+/dx$ and $d\mathcal{E}_-/dx$,

$$\exp\left(i\int_0^x k dx'\right)\frac{d}{dx}(\alpha^{1/2}\mathcal{E}_+)-\exp\left(-i\int_0^x k dx'\right)\frac{d}{dx}(\alpha^{1/2}\mathcal{E}_-)=-\kappa\alpha^{1/2}\mathcal{E}_+\exp\left(i\int_0^x k dx'\right)-\kappa\alpha^{1/2}\mathcal{E}_-\exp\left(-i\int_0^x k dx'\right)$$

$$-\frac{\omega_p^{\,2}}{\omega}\alpha^{-1/2}\int_{\Phi(x)}^{\infty} dE\frac{\partial F}{\partial E}\exp\left(2i\omega\int_x^{x_0}\frac{dx''}{v(x'')}\right)\int_{-\infty}^{x_0}dx'\,\mathcal{E}(x')\exp\left(i\omega\int_{x'}^{x}\frac{dx''}{v(x'')}\right)+O(\mathcal{E}_\pm''), \tag{4}$$

where

$$\alpha(x) \equiv -\frac{\omega_p^{\,2}}{\omega}\,\mathrm{Re}\int_{\Phi(x)}^{\infty} dE\,\frac{\partial F}{\partial E}\left[\frac{1}{(k-\omega/v)^2}-\frac{1}{(k+\omega/v)^2}\right], \tag{5}$$

$$\kappa(x) = -\frac{\pi\omega_p^{\,2}}{\omega\alpha}\int_{\Phi(x)}^{\infty} dE\frac{\partial F}{\partial E}\left[\delta\left(k-\frac{\omega}{v}\right)+\delta\left(k+\frac{\omega}{v}\right)\right], \tag{6}$$

and in Eq. (4) we have dropped small imaginary terms additive to α. In Eq. (4) the term $O(\mathcal{E}_\pm'')$ indicates terms formally of order δ^2 which are the result of the integration by parts. The contribution of these terms to the reflection coefficient has already been calculated,[5] so we will not consider them here.

In order to obtain separate equations for $\mathcal{E}_+$ and $\mathcal{E}_-$ from Eq. (4), we invoke an averaging process and equate to 0 separately the coefficients of the rapidly varying phase factors $\exp(\pm i\int_0^x k dx')$. The integral with respect to E on the right-hand side of Eq. (4) will contribute to $\mathcal{E}_+$ or $\mathcal{E}_-$ only when the phase factor in the integrand is the same as that of $d\mathcal{E}_+/dx$ or $d\mathcal{E}_-/dx$, respectively. For the chosen monotonic character of $\Phi(x)$, this situation does not occur for $\mathcal{E}_+$, so we obtain

$$\mathcal{E}_+(x) = \mathcal{E}_+(0)\left(\frac{\alpha(0)}{\alpha(x)}\right)^{1/2}\exp\left(-\int_0^x \kappa dx'\right) \tag{7}$$

in agreement with previous authors.[5,6] The substitution of Eq. (7) into the E integral on the right-hand side of Eq. (4) will give a contribution which has the same phase factor as $d\mathcal{E}_-/dx$, thus

$$\frac{d}{dx}(\alpha^{1/2}\mathcal{E}_-)=\kappa\alpha^{1/2}\mathcal{E}_- +\frac{\omega_p^{\,2}}{\omega}\alpha^{-1/2}\int_{\Phi}^{\infty} dE\frac{\partial F}{\partial E}\exp 2i\left(\omega\int_x^{x_0}\frac{dx''}{v(x'')}+\int_0^x k dx''\right)$$

$$\times\int_{-\infty}^{x_0}dx'\,\mathcal{E}_+(x')\exp i\left(\omega\int_{x'}^{x}\frac{dx''}{v(x'')}+\int_x^{x'}k dx''\right). \tag{8}$$

To show that the final term of Eq. (8) has the claimed slow spatial variation, we perform the x' integration approximately as

$$\int_{-\infty}^{x_0} dx' \,\mathcal{E}_+(x') \exp i\left(\omega\int_{x'}^{x} \frac{dx''}{v(x'')} + \int_{x}^{x'} k dx''\right) \approx \frac{2\kappa}{(\omega/v-k)^2+\kappa^2}\,\mathcal{E}_+(x), \tag{9}$$

where from Eq. (7) we have taken the dominant x dependence of $\mathcal{E}_+(x)$ to occur through the damping decrement κ (to be justified later) and chosen Imω so as to give convergence of the x' integration in each direction away from the point x. Note that where $v=\omega/k$, the phase factor in Eq. (8) is slowly varying in x. Because $F(E)$ in Eq. (8) is slowly varying in E compared with the peaked function of Eq. (9), it may be evaluated at $E=\frac{1}{2}(\omega/k)^2+\Phi$. Using this and Eq. (9), Eq. (8) may be integrated to give

$$\frac{\mathcal{E}_-(0)}{\mathcal{E}_+(0)} = \int_0^\infty dx \exp\left(2i\int_0^x k dx''\right)\left\langle \exp\left(2i\omega\int_x^{x_0}\frac{dx''}{v(x'')}\right)\right\rangle \frac{d}{dx}\exp\left(-2\int_0^x \kappa dx''\right), \tag{10}$$

where

$$\left\langle \exp\left(2i\omega\int_x^{x_0}\frac{dx''}{v(x'')}\right)\right\rangle \equiv \frac{1}{\pi}\frac{\partial}{\partial E}\left(\frac{\omega}{v}\right)_{v=\omega/k}\int_{\Phi(x)}^\infty dE\frac{\kappa}{(\omega/v-k)^2+\kappa^2}\exp\left(2i\omega\int_x^{x_0}\frac{dx''}{v(x'')}\right). \tag{11}$$

We wish to evaluate approximately the integrals appearing in Eqs. (10) and (11). Of the three functions appearing in the x integrand of Eq. (10), the third is sharply peaked relative to the product of the first two. This latter product varies on a scale $k^{-1}\delta$ whereas the peaked function varies on a scale $(\omega^2/k^2V_T^2)k^{-1}\delta$ around the maximum x_m, where

$$d\kappa/dx\,|_{x=x_m} = 2\kappa^2(x_m), \tag{12}$$

and V_T is the electron thermal velocity. Thus we obtain approximately for Eq. (10)

$$\frac{\mathcal{E}_-(0)}{\mathcal{E}_+(0)} = -\exp\left[2i\int_0^{x_m}(k+i\kappa)dx'\right]\left\langle\exp\left(2i\omega\int_x^{x_0}\frac{dx''}{v(x'')}\right)\right\rangle_{x=x_m} \tag{13}$$

From Eq. (6) and the scaling following Eq. (2), we see from Eq. (12 that

$$(\kappa k^{-1})_{x=x_m} \sim (\omega^2/k^2V_T^2)\delta \sim \delta\ln\delta$$

for small δ, justifying our assumptions that the damping was weak and that the dominant spatial variation of $\mathcal{E}_\pm(x)$ occurred through $\exp(\pm\int_0^x \kappa dx')$.

The smallness of $\kappa(x_m)$ is useful also in evaluating the E integral in Eq. (11). Let $L_m = x_0 - x_m$, i.e., the distance to the point of reflection from x_m. If $k(x_m)L_m \ll (\delta\ln\delta)^{-1}$, the exponential in the integrand of Eq. (11) is slowly varying in E relative to the peaked function and the integral becomes approximately

$$\left\langle\exp\left(2i\omega\int_x^{x_0}\frac{dx''}{v(x'')}\right)\right\rangle = \exp\left(2i\omega\int_x^{x_0}\frac{dx''}{v(x'')}\right)_{E=\frac{1}{2}(\omega/k)^2+\Phi(x)}$$

to be evaluated at $x=x_m$. This is just the exponential of the phase shift into and out of the sheath region for particles traveling at $v=\omega/k(x_m)$ at x_m. When $k(x_m)L_m \sim (\delta\ln\delta)^{-1}$ or larger, this perfect reflection will be reduced by phase mixing. However, even in this case, this effect may dominate the reflection coefficient based on the $O(\mathcal{E}_\pm'')$ terms in Eq. (4).

When we allow for reflection at an angle to the sheath through the inclusion of $k_y \neq 0$ and use as variables the perturbed potentials $\varphi_\pm$ rather than $\mathcal{E}_\pm$, our result, Eq. (8), is changed to read

$$\frac{d}{dx}(\alpha^{1/2}\varphi_-) = \kappa\alpha^{1/2}\varphi_- + \frac{2\omega_p^2\omega}{\alpha^{1/2}}\int d^3v\frac{\partial F}{\partial E}\exp 2i\left[(\omega-k_yv_y)\int_x^{x_0}\frac{dx''}{v(x'')}+\int_0^x k_x dx''\right]\frac{\kappa v_x}{(\omega-k_yv_y-k_xv_x)^2+\kappa^2v_x^2}. \tag{14}$$

Treating this equation in a manner similar to the above will yield additional phase mixing due to k_y.

Equations (3), (5), and (6) are modified as in Ref. 5.

The mechanism of reflection at the plasma sheath should be useful in further clarifying the description of Tonks-Dattner resonances in a positive column, for which Leavens has obtained numerical solutions to the Vlasov equation.[7] As properties of plasma waves in collisionless, nonuniform media, there may be applications of this and similar effects of the static electric field to certain solar and astrophysical phenomena, such as the plasma oscillations thought to give rise to type-III solar radio bursts,[8] the properties of the interaction of the solar wind with the ionized gas around the earth,[9] and $\vec{k}\cdot\vec{B}_0 \neq 0$ loss-cone instabilities in the magnetosphere.

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