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POSSIBILITY OF AN INFINITE SEQUENCE OF REGGE RECURRENCES

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We show that a Regge trajectory, $\alpha(s)$, cannot have the property $\mathrm{Re}\alpha(s)\to+\infty$ as $s\to+\infty$ without leading to inconsistencies with two features of dispersion theory and Regge pole theory: that both $\alpha(s)$ and the reduced residue function, $\gamma(s)$, are analytic in the cut plane with one cut, and that they and the partial wave amplitude, $a(l,s)$, for $\mathrm{Re}l=-\frac{1}{2}$, are bounded for large $|s|$ by $\exp[|s|^{\frac{1}{2}-\epsilon}]$.

Recently, there have been several papers[1,2] that dealt with Regge trajectories that approach infinity as $s\to+\infty$ and thus lead to an infinite sequence of Regge recurrences. To this purely theoretical interest one can add one experimental fact. Some of the known resonances, if interpreted as Regge recurrences, seem to lead to Regge trajectories that, up to the energies studied, increase linearly with s.[3]

It is almost obvious that if we have a Regge trajectory, $\alpha(s)$, such that $\mathrm{Re}\alpha(s)\to+\infty$ as $s\to+\infty$, the Mandelstam representation with a finite number of subtractions can no longer be valid for any scattering process in which such a trajectory can contribute to any channel. The introduction of such trajectories in more than one channel will also make the derivation of a crossing-symmetric Regge representation, given earlier[4] for the usual case, not feasible. In fact, if trajectories do not turn back for large s, t, and u, it would seem at first sight that there would be a basic contradiction between crossing symmetry and Regge behavior unless we impose strong conditions on the residue functions as $s\to\infty$. For example, in the model given in Ref. 1, Van Hove sums up an infinite number of single-particle exchange terms in the t channel to obtain a Regge term proportional to $P_{\alpha(t)}[1+2s/(t-4)]$. For large s this term leads to the correct Regge behavior. However, in a crossing-symmetric case one could also sum an infinite number of single-particle terms in the s channel. This would give a term proportional to $P_{\alpha(s)}[1+2t/(s-4)]$. The behavior of this term for large s will depend on how $\mathrm{Re}\alpha(s)\to+\infty$ as $s\to+\infty$ and on how fast the residue functions can decrease with s. It is not a priori obvious that this s-channel term could not be bigger than t-channel term as $s\to+\infty$.

The question naturally arises under what conditions can such trajectories exist? In this short note we show that they are not even consistent with some of the most common features of dispersion relations and Regge-pole theory. These features are the following:

(i) For fixed finite t, the scattering amplitude $f(s,t)$ is analytic in the cut s plane, and is bounded by a polynomial in $|s|$ as $|s|\to\infty$.

(ii) At fixed physical c.m. angle in the s channel, $\cos\theta=z$, $-1<z<+1$, $f(s,\cos\theta)/s^N\to0$ as $s\to+\infty$.

(iii) The partial-wave amplitude $a(l,s)$ satisfies the necessary conditions for the validity of the Sommerfeld-Watson transformation. We also assume, for l not near a Regge pole,

$$a(l,s)/s^N\to0 \text{ as } s\to+\infty.$$

We can even weaken (i)-(iii) considerably without affecting our main results. We can replace s^N by $\exp(|s|^{\frac{1}{2}-\epsilon})$ and still our conclusions will be the same.

To start the problem let us start with the usual Watson-Sommerfeld transformation for equal-mass scattering. We write

$$f(s,z)=f_B(s,z)+\frac{(2\alpha+1)\beta(s)}{\sin\pi\alpha(s)}[P_{\alpha(s)}(-z)\pm P_{\alpha(s)}(z)], \tag{1}$$

where we have assumed for simplicity that we have only one Regge pole in the s channel. Now for fixed physical z, $0<z<1$, and $s>4$ the integral defining the background term $f_B(s,z)$ converges absolutely. Using (iii) we see that

$$\lim_{s\to+\infty}\left|\frac{f_B(s,z)}{s^N}\right|=0,\quad 0<z<1. \tag{2}$$

It follows from (1) that for fixed $s>4$, $f_B(s,z)$ is an analytic function of z regular in the cut z plane, that $f_B(s,z)\sim|z|^{-1/2}$ as $|z|\to\infty$. We can then write an unsubtracted dispersion relation in z at fixed s for f_B and conclude that the discontinuities across the z cuts when divided by s^N also vanish as $s\to+\infty$. We use this dispersion relation to define $f_B(s,z)$ for unphysical z and also f_B as a function of s and t, where $z=1+2t/(s-4)$. It is easy to see then that[5]

$$\lim_{s\to+\infty}\left|\frac{f_B(s,t)}{s^N}\right|=0,\quad t\text{ fixed};$$

$$\lim_{s\to+\infty}\left|\frac{f_B(s,z)}{s^N}\right|=0,\quad \operatorname{Im}z\neq 0. \tag{3}$$

At fixed t, we have

$$f(s,t)=f_B(s,t)+\frac{\beta(s)(2\alpha+1)}{\sin\pi\alpha(s)}\left[P_{\alpha(s)}\left(-1-\frac{2t}{s-4}\right)\pm P_{\alpha(s)}\left(1+\frac{2t}{s-4}\right)\right]. \tag{4}$$

Both $f(s,t)/s^N$ and $f_B(s,t)/s^N$ vanish as $s\to\infty$. Then regardless of how $\alpha(s)$ behaves as $s\to+\infty$, we must satisfy the condition

$$\lim_{s\to+\infty}\left|\frac{\beta(s)(2\alpha+1)}{\sin\pi\alpha(s)}\left\{P_{\alpha(s)}\left(-1-\frac{2t}{s-4}\right)\pm P_{\alpha(s)}\left(1+\frac{2t}{s-4}\right)\right\}\right|s^{-N}=0, \tag{5}$$

for fixed t, $\operatorname{Re}t>0$ and $\operatorname{Im}t\neq 0$. This condition is trivially satisfied if $\operatorname{Re}\alpha(s)<-\frac{1}{2}$ for large positive s. The problem is however that if $\operatorname{Re}\alpha(s)\to+\infty$ as $s\to+\infty$ then we shall see below that it is impossible to satisfy (5) without giving up some of the properties of $\beta(s)$ and $\alpha(s)$ that are part of Regge-pole phenomenology.

We shall assume that, in addition to (i)-(iii), $\alpha(s)$ and $\beta(s)$ have the same analyticity properties proved in potential scattering[6] and at least heuristically made plausible in field theory,[7] namely:

(iv) We take $\alpha(s)$ to be a real analytic function of s with one cut from $s=4$ to $s=+\infty$, and assume $\alpha(s)$ to be bounded by a polynomial as $|s|\to\infty$ in all directions.

Furthermore, the reduced residue $\gamma(s)$ defined by

$$\gamma(s)=\beta(s)/(s-4)^{\alpha(s)}, \tag{6}$$

is also a real analytic function with a normal right-hand cut. Asymptotically $\gamma(s)$ is bounded by a polynomial in $|s|$ in all directions. (Again, it actually suffices for this paper to assume that $|\gamma(s)|<c\exp(|s|^{\frac{1}{2}-\epsilon})$ as $|s|\to\infty$ in all directions.)

We must stress here that the properties implied in (iv) will not hold if two trajectories cross for some s and we will have a strange branch point at that value of s.[6]

Our first step is to show that it follows from (5) and (iv) that if $\operatorname{Re}\alpha(s)\to+\infty$ as $s\to+\infty$ it could only do that slower than $s^{1/2}$. In fact we shall prove that in all directions in the s plane the quantity $|\alpha(s)|\ln|s|/|s|^{1/2}\to 0$ as $|s|\to\infty$. Once we are limited to trajectories that grow slower than $\sqrt{s}$ we show that such trajectories can have $\operatorname{Re}\alpha(s)\to+\infty$ as $s\to+\infty$ only if they also have $\operatorname{Re}\alpha(s)\to+\infty$ as $s\to-\infty$. Such trajectories are physically excluded since they would lead to an infinite number of ghosts on the negative real s axis, and they give a differential cross section for the t-channel reaction which increases for large momentum transfers. We recall that $s<0$ is a physical momentum transfer for the t-channel reaction.

Returning to (5) and using the relation between $P_\alpha(-z)$ and $P_\alpha(z)$, it turns out to be sufficient to study the condition

$$\lim_{s\to+\infty}\left|\gamma(s)(s-4)^{\alpha(s)}P_{\alpha(s)}\left(1+\frac{2t}{s-4}\right)\right|s^{-N}=0. \tag{5$'$}$$

If $\operatorname{Re}\alpha(s)\to+\infty$ as $s\to+\infty$ then P_α in (5′) could blow up exponentially. In fact starting with the integral representation

$$P_\alpha(z)=\frac{1}{\pi}\int_0^\pi[z+(z^2-1)^{\frac{1}{2}}\cos x]^\alpha dx,\quad \mathrm{Re}z>0; \tag{7}$$

we obtain as $s\to\infty$

$$P_{\alpha(s)}\left(1+\frac{2t}{s-4}\right)\cong I_0\left(\frac{2\alpha\sqrt{t}}{(s-4)^{1/2}}\right),\quad \mathrm{Re}t>0. \tag{8}$$

If $\alpha(s)/\sqrt{s}$ is large as $s\to+\infty$, we have

$$P_{\alpha(s)}\left(1+\frac{2t}{s-4}\right)\cong\left(\frac{4\pi\alpha(s)\sqrt{t}}{\sqrt{s}}\right)^{-1/2}\exp\left(\frac{2\alpha\sqrt{t}}{\sqrt{s}}\right). \tag{9}$$

To have any chance then to satisfy (5′), we must make the reduced residue $\gamma(s)$ fall off rapidly as $s\to\infty$. However, unfortunately this rate of fall of $\gamma(s)$ is already limited by the analyticity and boundedness condition in (iv). In fact one can use a theorem of Boas[8] to show that it follows from (iv) that

$$\int_4^\infty\frac{|\ln|\gamma(s)||}{(s-4)^{1/2}\cdot s}ds<\infty. \tag{10}$$

There must therefore exist at least an infinite sequence of intervals extending out to infinity such that

$$|\gamma(s_n)|>C\exp(-\epsilon s_n^{1/2}/\ln s_n), \tag{11}$$

where s_n is large and in the nth interval $s_n\to\infty$ as $n\to\infty$. The theorem we have used here is exactly the same one used by Martin[9] to study the fastest possible fall-off of form factors.

Substituting (11) and (9) in (5′) we see that (5′) can only hold if

$$\lim_{n\to\infty}\frac{\alpha_r(s_n)\ln^2 s_n}{s_n^{1/2}}=0, \tag{12a}$$

and

$$\lim_{n\to\infty}\frac{\alpha_i(s_n)(\ln s_n)}{s_n}=0. \tag{12b}$$

The latter limit on $\mathrm{Im}\alpha$ can be improved by going through the same argument at some fixed z, $\mathrm{Im}z\neq0$. One gets[10]

$$\lim_{n\to\infty}\frac{\alpha_i(s_n)\ln s_n}{s_n^{1/2}}=0. \tag{12b′}$$

These limits hold for an infinite sequence of intervals on the real axis[11] and unless we are willing to admit pathologically oscillating trajectories we must conclude that

$$\lim_{s\to\infty}\left|\frac{\alpha(s)}{\sqrt{s}}\ln s\right|=0. \tag{13}$$

Now $\alpha(s)$ is a real analytic function and thus (13) holds both above and below the cut on the physical sheet. We can use the Phragmén-Lindelöf theorem to show that in all directions on the physical sheet of the s plane, we have

$$\lim_{|s|\to\infty}\left|\frac{\alpha(s)\ln|s|}{|s|^{1/2}}\right|=0. \tag{14}$$

So far we have only shown that trajectories can only tend to $+\infty$ slower than $\sqrt{s}$. Clearly (14) should not be used for trajectories for which $\mathrm{Re}\alpha(s)\to-\infty$ as $s\to\infty$.

We proceed to the second part of our assertion. We show below that $\mathrm{Re}\alpha(s)$ cannot tend to $+\infty$ as $s\to+\infty$ no matter how slowly, unless also $\mathrm{Re}\alpha(s)\to+\infty$ as $s\to-\infty$. For let us assume that $\mathrm{Re}\alpha(s)\to+\infty$ as $s\to+\infty$. Let us furthermore exclude the physically uninteresting case where $\mathrm{Re}\alpha(s)$ keeps on oscillating for large positive s with amplitudes increasing to infinity. Then there must exist a point $s=s_0$ such that

$$\mathrm{Re}\alpha(s)>0,\quad s>s_0. \tag{15}$$

We define the function $\alpha'(s)$ as

$$\alpha'(s)=\alpha(s)+c,\quad c>0, \tag{16}$$

where c is given by

$$-c=\min_{4\leq s\leq s_0}[\mathrm{Re}\alpha(s)]. \tag{17}$$

Hence by construction $\alpha'(s)$ is a real analytic function and on both sides of the cut $\mathrm{Re}\alpha'(s)>0$. (Clearly if $s_0<4$ we would have the same result with $c=0$ and $\alpha'=\alpha$.) Indeed one sees that since $\mathrm{Re}\alpha'$ is a harmonic function which is positive on the boundary of the physical sheet of the s plane and with growth at infinity limited as in (14), then $\mathrm{Re}\alpha'$ must be positive for all s in the physical sheet. We can write a Poisson representation for $\mathrm{Re}\alpha'$. For this it is more convenient to use the variable $k=\frac{1}{2}(s-4)^{1/2}$ and map the physical sheet of the s plane onto the upper half k plane. We have

$$k=x+iy,$$
$$\alpha'(k)=u(x,y)+iv(x,y), \quad y\geq 0. \tag{18}$$

The real part of α', u, now admits the representation

$$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{u(x',0)}{(x'-x)^2+y^2}\,dx', \quad y\geq 0. \tag{19}$$

where $u(x',0)=\mathrm{Re}\alpha'(s)>0$ by construction. It follows that $u(x,y)>0$ for all x and $y>0$. No subtraction terms are necessary in (19) because of the condition (14) and the integral is absolutely convergent. The function $u(x',0)$ is an even function of x', and for $x=0$, (19) becomes

$$u(0,y)=\frac{2y}{\pi}\int_0^{\infty}\frac{u(x',0)}{x'^2+y^2}dx'. \tag{20}$$

If the trajectory goes to infinity no matter how slowly there must be a point s_L such that for $s>s_L$, $\mathrm{Re}\alpha(s)>M$ with M being some large positive number. For $s>s_L$, $\mathrm{Re}\alpha'>(M+c)$ and hence $u(0,y)$ has the lower bound

$$u(0,y)>\frac{2y}{\pi}\int_{X_L}^{\infty}(M+c)\frac{dx'}{x'^2+y^2}$$
$$=(M+c)\left(1-\frac{2}{\pi}\tan^{-1}\frac{X_L}{y}\right). \tag{21}$$

For sufficiently large y, $u(0,y)>M$. This means that any trajectory that satisfies (14) and for which $\mathrm{Re}\alpha(s)$ goes to $+\infty$ as $s\to+\infty$, $\mathrm{Re}\alpha(s)$ must also go to $+\infty$ as $s\to-\infty$. If $\mathrm{Re}\alpha(s)\to+\infty$ as $s\to-\infty$ then we have a trajectory that has an infinite number of Regge ghosts which appear every time $\mathrm{Re}\alpha(s)$ crosses a positive integer value for negative real s. But there is clearly a more important physical reason why such a trajectory cannot be accepted. Negative real s is a physical momentum transfer for the t-channel reaction. If $\mathrm{Re}\alpha(s)\to+\infty$ as $s\to-\infty$ then the differential cross section for the t-channel reaction at large momentum transfers would become arbitrarily large and bypass all bounds.

We summarize our results by stating that if a trajectory exists for which $\mathrm{Re}\alpha(s)\to+\infty$ as $s\to+\infty$, and which does not go to $+\infty$ as $s\to-\infty$, then at least one of the following three statements must be false: First, the statement that $\gamma(s)$ and $\alpha(s)$ have cut-plane analyticity with only a right-hand cut; second, the statement that $a(l,s)$ for $l=-\frac{1}{2}+i\lambda$, $\gamma(s)$, and $\alpha(s)$ are all bounded by $\exp[|s|^{\frac{1}{2}-\epsilon}]$ as $|s|\to\infty$ in all directions on the physical sheet; and third, the statement that $f(s,t)$ and $f(s,z)$ are bounded for large s by at least $\exp(|s|^{\frac{1}{2}-\epsilon})$. We of course use the Regge analyticity of $a(l,s)$ as a function of l and the asymptotic conditions in l necessary for the validity of the Watson-Sommerfeld formula. But without that the whole game is up.

Finally, we must remark on the fact that we took only one trajectory in (1). One can legitimately ask if it cannot happen that we might have, say, two trajectories $\alpha_1(s)$ and $\alpha_2(s)$ both having $\mathrm{Re}\alpha(s)\to+\infty$ as $s\to+\infty$ but in such a way that the contributions from the two trajectories cancel each other as $s\to+\infty$ and (5) is satisfied for the sum of two Regge terms but not for each alone. However, to get any such significant cancellation $[\alpha_1(s)-\alpha_2(s)]$ must vanish faster than $\exp[-s^{1/2}\ln^{-1}s]$ as $s\to\infty$. This rate of decrease is not allowed by the theorem of Ref. 8. One way, nevertheless, remains open to get cancellations, at least mathematically. This could occur if we have an infinite set of trajectories, $\alpha_n(s)$, $n=0,1,2,\cdots$, all having $\mathrm{Re}\alpha_n(s)\to+\infty$ as $s\to+\infty$ and with the residues all "cooked up" to give a super cancellation as $s\to\infty$. In that case (5) would be replaced by an infinite sum of similar terms over all the α_n's. One might speculate that the set $\alpha_n(s)$ is nothing but the daughter trajectories of Freedman and Wang.[12] To go further at present would be highly speculative.

In conclusion we must state that if trajectories do actually go to positive infinity as $s\to+\infty$, then if we do not want to give up the weak bounds we placed on $a(l,s)$, $\gamma(s)$, and $\alpha(s)$, it could be that trajectories cross each other for some s. This as we mentioned earlier will lead to new branch points in $\gamma(s)$ and $\alpha(s)$ which will make (iv) invalid.

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Note added in proof.—It would perhaps clarify our paper if we mention that our results remain essentially unchanged if we use the residue $\beta(s)$ all the way and do not use $\gamma(s)$.

In the equal-mass case $\beta(s)$ is also analytic with one cut. The same theorem of Boas can be used for $\beta(s)$ if it is bounded by $\exp[|s|^{\frac{1}{2}-\epsilon}]$. This tells us that $\beta(s)$ cannot fall off faster than $\exp[-\epsilon s^{1/2}\ln^{-1}s]$. We can choose to work at fixed complex z instead of fixed t. In that case one has, instead of (5′), the condition

$$\lim_{s\to\infty} |\beta(s)P_{\alpha(s)}(z)|s^{-N} = 0.$$

This gives the same result as (12a) with one power of $\ln s$ less.

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[5]This argument would be more rigorous if we use instead of (1) a modified Regge representation which exhibits the correct cuts in the z plane starting at $z=1+8/(s-4)$. See, for example, N. N. Khuri, Phys. Rev. 130, 429 (1963). The part of the background term used in that paper to modify the Regge-pole contribution cannot give contributions for large s that will invalidate Eq. (5) below.

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[10]This result can also be obtained at fixed t by letting $s\to+\infty$ below the cut.

[11]In fact for large s the sequence of intervals on which (11) does not hold must get narrower as $s\to\infty$ in such a way that $\int_S ds\,(s\log s)^{-1}$ converges. Here S denotes the union of these intervals.

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NONEXISTENCE OF SELF-CONJUGATE PARTICLES WITH HALF-INTEGRAL ISOSPIN*

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Recently it has been shown by Carruthers,[1] in the framework of local field theory, that the field operators corresponding to spinless bosons of a self-conjugate multiplet with half-integral isospin (SMHI) are nonlocal, in the sense that local commutativity between fields φ and $\varphi^\dagger$ cannot be satisfied. We shall generalize this result to particles of any spin in an SMHI. The conclusion is that the requirement of local commutativity and the group structure of SU(2) do not allow us to construct spin fields of such particles. An analogous result is proved in the analytic S-matrix framework, where the requirement of isospin invariance plus the usual crossing property entail that all scattering amplitudes involving any particles of an SMHI must vanish. Interesting physical implications of this result, and generalizations to higher internal symmetry, are also discussed.

Throughout this paper, by a self-conjugate multiplet we mean an irreducible multiplet that contains the antiparticle of each particle contained in the multiplet. Consider a self-conjugate isomultiplet of spin j and isospin I. Let $a_\alpha^\dagger(\vec{p},\sigma)$ and $a_\alpha(\vec{p},\sigma)$ be the creation and annihilation operators of the free-particle multiplet with momentum $\vec{p}$ and spin component σ, where α denotes the I_z component. Isospin symmetry is expressed by

$$U^{-1}(u)a_\alpha(\vec{p},\sigma)U(u)=\sum_{\alpha'}D_{\alpha\alpha'}^{(I)}(u)a_{\alpha'}(\vec{p},\sigma), \quad (1)$$

where $U(u)$ is the unitary operator in Hilbert space that represents the SU(2) transformation u, and $D^{(I)}(u)$ is the standard irreducible representation matrix[2] with dimension $(2I+1)$. The adjoint of (1) is

$$U^{-1}(u)a_\alpha^\dagger(\vec{p},\sigma)U(u)=\sum_{\alpha'}D_{\alpha\alpha'}^{(I)*}(u)a_{\alpha'}^\dagger(\vec{p},\sigma). \quad (2)$$

Now we construct the $(2j+1)$-component field in the usual way[3]:

$$\varphi_\sigma^{\ \alpha}(x)=(2\pi)^{-\frac{3}{2}}\int\frac{d^3p}{(2p_0)^{1/2}}\sum_{\sigma'}[D_{\sigma\sigma'}^{(j)}[L(\vec{p})]a_\alpha(\vec{p},\sigma')e^{ip\cdot x}+\{D^{(j)}[L(\vec{p})]C^{-1}\}_{\sigma\sigma'}\{Ha^\dagger(\vec{p},\sigma')\}_\alpha e^{-ip\cdot x}], \quad (3)$$