GRAVITATIONAL-RADIATION DETECTION RANGE FOR BINARY STELLAR SYSTEMS

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Among the possible types of astronomical sources of gravitational radiation are binary stellar systems. Conceptually, these can be two quasistellar bodies with masses of 10^6 to 10^8 $M_\odot$,[1] a pair of neutron stars,[2] or a small planetary-sized object falling into a collapsed star.[3] The power level of radiation from these sources can, in principle, be extremely high. A number of estimates of the typical radiation power levels and frequencies and the prospect of detecting the radiation from such systems have been discussed in the literature[1-12]; however, these estimates were usually based on simplified models and have tended to be contradictory.

The most general description of the characteristics of the gravitational radiation from binary systems and the behavior of the systems as they radiate can be found in the papers by Peters and Mathews[4] and Peters.[5] These analyses come from expansions of the field equations of general relativity in powers of the gravitational coupling constant; they accurately describe the gravitational interaction, but assume point masses and subrelativistic velocities of the components.

Detectors for gravitational radiation basically consist of an extended mass-spring system which interacts with the differential forces induced by the dynamic gravitational-force gradient fields in the radiation. The basic principles of gravitational radiation detectors were first derived by Weber,[6] and a discussion of these detectors and their interaction with various sources can be found in papers by Weber[6,7,12] and others.[8-10]

In this paper we briefly summarize our investigations of the characteristics of the gravitational radiation to be expected from binary systems, and the manner in which this radiation interacts with the present gravitational antenna designs. We have found that the behavior of such systems is adequately described by the subrelativistic equations of Peters and Mathews,[4,5] even for frequencies to 10 kHz and power levels to 10^{48} W, provided the masses are sufficiently dense; in addition, we have found that the theoretical maximum power output of a binary system is independent of the total or relative masses of the components. Our studies also indicate that the maximum gravitational-radiation detection range for binary stellar systems by mechanically resonant antennas is independent of the type of binary system and that for maximum detection range it is desirable to utilize a wide-bandwidth detector.

The time-averaged gravitational radiation power emitted by a binary system of low eccentricity with masses m and M separated by a distance a is given by[5]

$$P(a)=\frac{32G^4m^2M^2(m+M)}{5c^5a^5}\equiv\gamma\frac{GmM}{8a^5}. \tag{1}$$

The angular rate of rotation of the system is[5]

$$\dot{\psi}^2=G(m+M)/a^3. \tag{2}$$

Because of the emission of energy and angular momentum in the form of gravitational radiation, the orbit decays and the orbital radius varies as a function of time,[5]

$$a^4(t)=\frac{256G^3mM(m+M)}{5c^5}t\equiv\gamma t. \tag{3}$$

Here t is taken to be the time to collapse of the system. The frequency of the gravitational radiation is at twice the rotation frequency because of the quadrupole nature of the gravitational radiation and is given as a function of time by

$$f=\frac{\dot{\psi}}{\pi}=\frac{G^{1/2}(m+M)^{1/2}}{\pi\gamma^{3/8}t^{3/8}}. \tag{4}$$

The power output is also a function of time,

$$P(t)=GmM/8\gamma^{1/4}t^{5/4} \tag{5}$$

(see Fig. 1); as the time to collapse approaches zero, the power output increases dramatically. This process is limited, however; at this high power output there can be only a finite number of rotations of the system before all of the mass quadrupole angular momentum is radiated away and the binary system collapses into a single axially symmetric spinning mass. The dashed portions of the curves in Fig. 1 indicate the region where we are violating the assumption of subrelativistic velocities; the structure at the top of the curves indicates

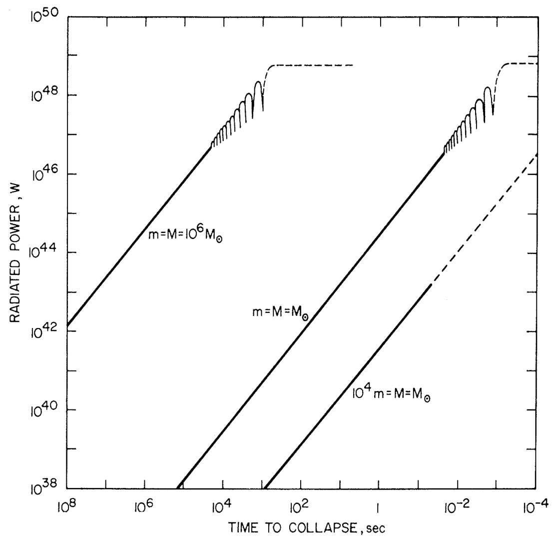

FIG. 1. Radiated gravitational power versus time to collapse for binary stellar systems.

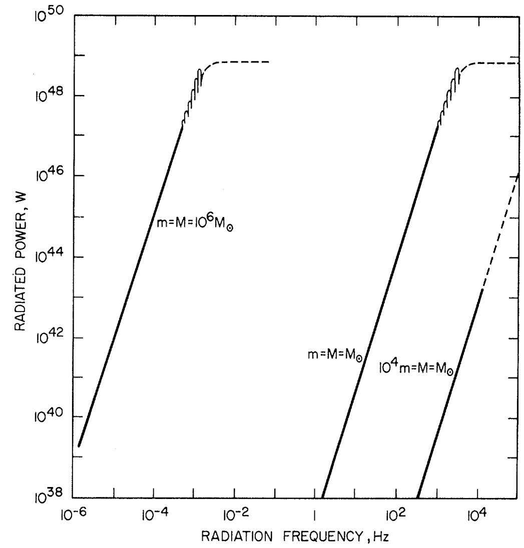

FIG. 2. Radiated gravitational power versus radiation frequency for binary stellar systems.

that although we are still well below relativistic velocities, the amplitude and frequency of the radiation are changing so rapidly that the definition of the "time-averaged power output" is beginning to lose its meaning. The number of radians of rotation before collapse can be obtained by integrating (2):

$$\psi = 8G^{1/2}(m+M)^{1/2}t^{5/8}/5\gamma^{3/8}. \qquad (6)$$

Combining (6) and (5), we obtain the interesting relation

$$P(\psi) = c^5/160G\psi^2. \qquad (7)$$

Equation (7) is independent of the total mass and mass ratio of the binary components and indicates that all binary systems have the same history of power output as a function of the number of radians to collapse.

The above equation is quite adequate for describing symmetric binary star systems of high density, since the assumptions of subrelativistic velocities ($v < c$) and linear gravitational fields ($a < GM/c^2$) start to break down only in the last 2π radians of rotation. (For a small component orbiting a larger component, these relativistic effects occur earlier in the collapse process.) The theoretical maximum power can be estimated from (7) by assuming that $\psi_{\rm min} = 2\pi$ and is found to be 6×10^{48} W. The systems' ability to attain such power outputs depends upon the components' ability to satisfy the assumption of point masses, since other energy loss mechanisms such as tidal deformation and mass flow have been neglected. A binary system containing neutron stars of the type described by Wheeler[13] satisfies the point-mass assumption quite well and forms a contact binary system only in the last cycle of rotation.

Although the peak power is the same for all systems, this does not mean that the total energy emitted is the same, or that the peak cannot occur at any frequency. The power as a function of frequency is easily obtained from (4) and (5) and is

$$P(f) = \frac{\pi^{10/3} mM\gamma f^{10/3}}{8G^{2/3}(m+M)^{5/3}}. \qquad (8)$$

This is plotted in Fig. 2 for several different sources. We see from this figure that as the masses orbit about each other, the energy and angular momentum of the system is radiated away as a "chirp" of gravitational radiation with a rapidly increasing power and frequency level. If we wish to detect these sources, we should design our antennas to operate at high frequencies where there is a significant amount of power being emitted.

The mean flux at the earth from one of these sources at a distance R is just

$$S = P/4\pi R^2. \qquad (9)$$

The actual flux will vary slightly about this mean, depending upon the orientation of the source with respect to the direction toward the earth.

For the discussion of the response of the gravitational antennas to this flux we shall use Weber's[6] Eq. (8.27). This equation describes the response of a resonant mass quadrupole to a sustained oscillatory gravitational-radiation field with a frequency at or near the antenna resonant frequency. Under these conditions, the power P_a absorbed by the antenna is a function of its capture cross section σ, which is a function of the mass μ, length l, frequency f_0, and mechanical quality factor Q of the antenna:

$$P_a = \sigma S = (15\pi G \mu l^2 f_0 Q / 4c^3)S. \tag{10}$$

However, an antenna can absorb only that radiation with a frequency within its bandwidth ($B = f_0/Q$); the binary stellar sources have such a rapidly changing frequency in the high-power regime that they stay within the bandwidth of the antenna for only a short time τ. The number of cycles emitted within the bandwidth can be calculated from (4) from the times t_1 and $t_1 + \tau$ when $f = f_0 - \frac{1}{2}B$ and $f_0 + \frac{1}{2}B$:

$$n = f_0 \tau = \frac{8G^{4/3}(m+M)^{4/3}}{3\pi^{8/3}\gamma Q f_0^{5/3}}. \tag{11}$$

Equation (11) indicates that if we operate in the high-frequency region we will detect only a small number of cycles of the radiation. In order that we do not violate the assumption implicit in the use of Weber's cross-section equation (i.e., that the radiation wave train is long compared with the antenna decay time), the minimum n we can use is $n = Q$.

The important factor in antenna design is the signal-to-noise power ratio. This is relatively easy to calculate for laboratory antennas since it has been found possible to eliminate all sources of noise from laboratory gravitational-radiation antennas except for the internal thermal noise.[12,14] (This is not true for the earth modes.[10]) For unity signal-to-noise power ratio we have

$$1 = P_a / kTB = P\sigma / 4\pi R^2 kTB, \tag{12}$$

where kT is the thermal noise energy and B is the instantaneous bandwidth.

If we now use $n = Q$ in (11) and rearrange, substitute for f_0 in the signal-to-noise Eq. (12), and rearrange again, we find that the range at which we can detect a source is given by

$$R_{\max}^2 = \frac{25c^2\mu l^2}{1536\pi kTQ^2}. \tag{13}$$

This equation indicates that the maximum range at which we can detect a binary stellar source does not depend upon any of the parameters of the source, but only on those of the antenna. It also indicates that for maximum detection range it is better to utilize a wide-bandwidth detector. This makes sense physically if we realize that the binary system is radiating significant amounts of power only in the last stages of the radiation process; at this point the frequency is shifting so rapidly that a wide-bandwidth detector is needed to capture a significant number of cycles of the radiation.

In the derivation of (13) we assumed that the bandwidth and frequency of the antenna are constant and related by $B = f_0/Q$. This does not hold for a detecting system with a swept center frequency or a chirp filter. A detector with these properties would be more suitable than a simple resonant mass quadrupole for the detection of binary systems. We have also assumed that the system parameters were chosen to provide for a fully developed antenna response, since this is the way the present gravitational antennas operate. Additional calculations have been made assuming that it was possible to obtain the signal-to-noise energy ratio in a time shorter than the mechanical response time of the antenna. In this case the maximum detection range does not have the $1/Q$ dependence, but instead depends upon the response time of the instrumentation. In the limit of instantaneous signal-to-noise measurement, a high-Q system would have the same maximum range as that predicted by (13) for low Q.

From (13) we can see that a large ($l = 1$ m), massive (1 ton) resonant antenna with a Q of 3 and a frequency in the kilocycle region could detect the gravitational radiation from a collapsing neutron-star binary system at 3000 light years.

Although there are roughly 10^8 observable stellar systems within 3000 light years, of which approximately 10^5 are binary systems with periods less than a day,[11] no neutron star has yet been identified,[15] much less a neutron-star binary system; therefore, it is not possible at the present time to estimate the frequency of occurrence of such an event, except to say that it is probably low.

[1]I. S. Shklovskii and N. S. Kardashev, Dokl. Akad. Nauk SSSR 155, 1039 (1964) [translation: Soviet Phys.—Doklady 9, 252 (1964)].

[2]F. J. Dyson, in Interstellar Communication, edited by A. G. W. Cameron (W. A. Benjamin, Inc., New York, 1963), Chap. 12. [Freeman Dyson has requested that attention be brought to a numerical error in Eq. (11) of this reference. The correct expression for power should be $W = 2048V^{10}/5c^5G$.]

[3]Ya. B. Zeldovich and I. D. Novikov, Dokl. Akad. Nauk SSSR 155, 1033 (1964) [translation: Soviet Phys.-Doklady 9, 246 (1964)].

[4]P. C. Peters and J. Mathews, Phys. Rev. 131, 435 (1963).

[5]P. C. Peters, Phys. Rev. 136, B1224 (1964).

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[7]J. Weber, in Gravitation and Relativity, edited by Hong-Yee Chiu and W. F. Hoffmann (W. A. Benjamin, Inc., New York, 1964), Chap. 5, p. 104, footnote.

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[11]V. N. Mironovskii, Astron. Zh. 42, 977 (1965) [translation: Soviet Astron.—AJ 9, 752 (1966)].

[12]J. Weber, Phys. Rev. Letters 18, 1967).

[13]J. A. Wheeler, Ann. Rev. Astron. Astrophys. 4, 393 (1966).

[14]J. Weber, Phys. Rev. Letters 17, 1228 (1966).

[15]Although neutron stars have not been seen, this does not mean they are nonexistent. Because of their small size, they have a low optical luminosity and would be unobservable at distances greater than a few light years (Ref. 13).

NONFORWARD SUPERCONVERGENCE RELATIONS

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A superconvergence relation for $\pi\rho$ scattering is saturated for a range of t with an infinite number of mesons of all spins and nondegenerate masses.

Recently Fubini derived the SU(6) results $g_{\varphi\rho\pi} = 0$ and $g_{\rho\pi\pi}{}^2 = \frac{1}{2} g_{\omega\rho\pi}{}^2 M^2$ by saturating two superconvergent sum rules (SCR) for $\pi\rho$ scattering at $t = 0$.[1] Fubini has also shown two ways by which an SCR could be saturated for a range of t without using states of isospin two: (1) The SCR can be saturated with a finite number of particles, but some coupling constants are forced to be imaginary; (2) the SCR can be saturated with an infinite number of particles (a tower), but they must be degenerate in mass.[2] We shall show that the SCR can be saturated for a range of t by using a tower of mesons not necessarily degenerate in mass.

As the saturating particles may have high spin, it is convenient to use helicity amplitudes. $T_{bb',aa'}{}^I(s,t)$ is the helicity amplitude for the t-channel process $(A + A' \rightarrow B + B')$. The superscript labels isospin, and the subscripts label helicity. $\overline{T}_{bb',aa'}{}^I(s,t)$ is the amplitude free of kinematical singularities as given by Wang.[3] Similarly $S_{a'b',ab}{}^I(s,t)$ is for the s-channel process $(A + B \rightarrow A' + B')$.

If we allow ourselves to treat ρ as a stable particle and if $T_{aa'}{}^1(s,t)/s \rightarrow 0$ as $s \rightarrow \infty$ for the reaction $\rho + \rho \rightarrow \pi + \pi$, then the fixed-$t$ dispersion relation for $\overline{T}_{+-}{}^1(s,t)$ can be converted into the SCR[4]

$$\int_{M_\pi{}^2}^{\infty} \mathrm{Im}\overline{T}_{+-}{}^1(s',t)ds' = 0. \qquad (1)$$

Equation (1) corresponds to the SCR for the invariant amplitude $A(s,t)$ used by Fubini.[1] The SCR which Fubini writes for $\overline{T}_{+-}{}^2(s,0)$, corresponding to the invariant amplitude $B(s,t)$, may not be correct because of the presence of Regge cuts,[5] and will not be considered in this note.

The amplitude $\overline{T}_{+-}{}^1(s,t)$ can be related to $S_{a,a'}{}^I(s,t)$ by using the helicity crossing matrices[6] and the isospin crossing matrix $C_{II'}$:

$$\overline{T}_{+-}{}^1(s,t) \equiv T_{+-}{}^1(s,t)[(1-x_t{}^2)(t-4\mu^2)]^{-1} = \sum_{a,a',I} (-)^{a'} C_{1I} S_{a',a}{}^I(s,t) \frac{d_{a',+}{}^1(x_A) d_{a,+}{}^1(x_A) M^2}{4sq^2(1-x_1{}^2)}, \qquad (2)$$

$$M = M_\rho, \quad \mu = M_\pi, \quad x_s = 1 + t/2q^2, \quad x_1 = \left[\frac{t}{t-4M^2}\right]^{1/2} \frac{E}{q}, \quad 1 - x_t{}^2 = \frac{(1-x_1{}^2)4sq^2}{M^2(t-4\mu^2)},$$