LOCAL CURRENT ALGEBRA AND MAGNETIC MOMENTS*

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The purpose of this Letter is to present a very simple calculation of the nucleon magnetic moments based on the local SU(3)⊗SU(3) algebra[1] and the angular-momentum condition of Dashen and Gell-Mann.[2]

To be precise, our assumptions are the following:

(A) The local SU(3)⊗SU(3) algebra:

$$[F_i(\vec{k}), F_j(\vec{k}')] = if_{ijk}F_k(\vec{k}+\vec{k}'), \quad [F_i(\vec{k}), F_j^5(\vec{k}')] = if_{ijk}F_k^5(\vec{k}+\vec{k}'), \quad [F_i^5(\vec{k}), F_j^5(\vec{k}')] = if_{ijk}F_k(\vec{k}+\vec{k}'). \tag{1}$$

(B) The angular condition: the matrix elements[2,3]

$$\langle N'h', P_x = -\tfrac{1}{2}k, P_y = 0, P_z = \infty\,|\exp\{-i\varphi J_y\}F_i(\vec{k})\exp\{-iJ_y\varphi\}|N, h, P_x = \tfrac{1}{2}k, P_y = 0, P_z = \infty\rangle \tag{2}$$

must have $\Delta J_x = 0, \pm 1$.

In Eq. (2),[4] $|Nh\rangle$ represents a definite helicity state (h) of the nucleon (N), φ is given by $\varphi = \arctan(k/2m_N)$, and J_y acts only on the helicity of the state.[2]

To satisfy conditions (A) and (B) we follow the suggestion of Gell-Mann[3] and represent the local current algebra by

$$F_i(\vec{k}) = U_1\tfrac{1}{2}\lambda_i^{(1)}e^{i\vec{k}\cdot\vec{x}_1}U_1^{-1} + U_2\tfrac{1}{2}\lambda_i^{(2)}e^{i\vec{k}\cdot\vec{x}_2}U_2^{-1} + U_3\tfrac{1}{2}\lambda_i^{(3)}e^{i\vec{k}\cdot\vec{x}_3}U_3^{-1}, \tag{3}$$

$$F_i^5(\vec{k}) = U_1\tfrac{1}{2}\lambda_i^{(1)}\sigma_z^{(1)}e^{i\vec{k}\cdot\vec{x}_1}U_1^{-1} + U_2\tfrac{1}{2}\lambda_i^{(2)}\sigma_z^{(2)}e^{i\vec{k}\cdot\vec{x}_2}U_2^{-1} + U_3\tfrac{1}{2}\lambda_i^{(3)}\sigma_z^{(3)}e^{i\vec{k}\cdot\vec{x}_3}U_3^{-1},$$

where the $\vec{x}_n$'s are the position operators of the quarks relative to the "center of mass" ($\vec{x}_1+\vec{x}_2+\vec{x}_3=0$); the $\vec{\sigma}^{(n)}$'s are the spin operators, and $\lambda_i^{(n)}$ are the matrices of the three-dimensional representation of SU(3). The U_n's are unitary operators to be chosen in such a way that Eqs. (1) and (2) are satisfied. In particular, this implies that

$$[U_1\vec{x}_1U_1^{-1}, U_2\vec{x}_2U_2^{-1}] = 0, \quad [U_1\sigma_z^{(1)}U_1^{-1}, U_2\sigma_z^{(2)}U_2^{-1}] = 0, \quad [U_1x_1U_1^{-1}, U_2\sigma_z^{(2)}U_2^{-1}] = 0, \tag{4}$$

etc. We then make the transformation to the "center-of-mass" frame and define

$$\vec{x}_1 = \vec{X} + (2\vec{x}/3), \quad \vec{x}_2 = \vec{X} - \tfrac{1}{3}\vec{x} - \tfrac{1}{2}\vec{x}', \quad \vec{x}_3 = \vec{X} - \tfrac{1}{3}\vec{x} - \tfrac{1}{2}\vec{x}'. \tag{5}$$

With $\vec{p}$, $\vec{p}'$, and $\vec{P}$ the momenta conjugate to $\vec{x}$, $\vec{x}'$, and $\vec{X}$, respectively, it is natural[3] to represent the angular-momentum operator in the $P_z = \infty$, $\vec{X} = 0$ frame by

$$\vec{J} = \tfrac{1}{2}\vec{\sigma}^{(1)} + \tfrac{1}{2}\vec{\sigma}^{(2)} + \tfrac{1}{2}\vec{\sigma}^{(3)} + \vec{x}\times\vec{p} + \vec{x}'\times\vec{p}'. \tag{6}$$

To calculate the magnetic moments, we need some information on the nucleon wave function. We will assume the following:

(C) At $P_z = \infty$, the nucleon which is made of three (real or mathematical) quarks is in an <u>s state</u> and belongs to the 56-dimensional representation of the (static) SU(6) group generated by λ_i and $\vec{\sigma}$. If the U_n's in Eq. (3) were equal to 1, this <u>56</u> would correspond to a pure representation of the algebra of charges and the anomalous magnetic moments would vanish <u>a priori</u>.[5] However, the U_n's cannot be equal to 1 due to the angular condition and, therefore, this "static" <u>56</u> is not a pure representation but a mixture[6] of representations of the "real" algebra of charges [generated by $F_i(0)$ and $F_i^5(0)$ as defined in Eq. (3)].

(D) Finally noticing that the angular condition, Eq. (2), can be written as

$$\langle N'h'|[J_x, [J_x, [J_x, \exp\{-i\varphi J_j\}F_i(\vec{k})\exp\{-iJ_y\varphi\}]]]|Nh\rangle = \langle N'h'|[J_x, \exp\{-i\varphi J_y\}F_i(\vec{k})\exp\{-i\varphi J_y\}]|Nh\rangle,$$

and is expansible in a power of $1/m_N$, we will assume that the following series makes sense[7] when sandwiched between nucleon states:

$$U_1x_1U_1^{-1}=x_1+(1/m_N)\nu_{1x}+\cdots,\quad U_2x_2U_2^{-1}=x_2+(1/m_N)\nu_{2x}+\cdots,$$
$$U_1\sigma_z^{(1)}U_1^{-1}=\sigma_z^{(1)}+(1/m_N)\Sigma_z^{(1)}+\cdots,\tag{7}$$

and so on.

The y component of the anomalous magnetic-moment operator, defined as the $M1$ part of $i(\partial/\partial k)(F_3+F_8/\sqrt{3})$ at $k=0$, is given in this scheme by

$$(\mu_A)_y=M1 \text{ part of } (-U_1x_1U_1^{-1}q^{(1)}-U_2x_2U_2^{-1}q^{(2)}-U_3x_3U_3^{-1}q^{(3)}),$$

where the $q^{(i)}$'s are the charges of the quarks.

Using the angular condition to order $1/m_N$ we have, as far as the spin part is concerned,[8] the following requirements on ν_{1x} and $\Sigma_z^{(1)}$ when sandwiched between infinite-momentum states:

$$\langle N'h'|\nu_{1x}-\tfrac{1}{2}(\sigma_y^{(1)}+\sigma_y^{(2)}+\sigma_y^{(3)})|N,h\rangle,$$
$$\langle N'h'|\Sigma_z^{(1)}|Nh\rangle,$$

and

$$\langle N'h'|\Sigma_z^{(1)}x_1+\sigma_z^{(1)}\nu_{1x}-\tfrac{1}{2}\sigma_z^{(1)}(\sigma_y^{(2)}+\sigma_y^{(2)})|Nh\rangle$$

must all have $\Delta J_x=0,\pm1$. This leads immediately, using Eq. (5), to[9]

$$\Sigma_z^{(1)}=-(3\alpha/2)(\sigma_x^{(1)}p_x+\sigma_y^{(1)}p_y),\quad \nu_{1x}=\tfrac{1}{2}(\alpha\sigma_y^{(1)}+\sigma_y^{(2)}+\sigma_y^{(3)}).\tag{8}$$

α is an arbitrary number which is not fixed by the angular condition. By symmetry we get analogous expressions for ν_{2x} and ν_{3x}. To keep the algebra requires, furthermore, by Eq. (4), that

$$[\Sigma_z^{(1)},x_2]+[\sigma_z^{(1)},\nu_{2x}]=0$$

or, by Eqs. (5) and (8)

$$[\Sigma_z^{(1)},-\tfrac{1}{3}x+\tfrac{1}{2}x']+[\sigma_z^{(1)},\tfrac{1}{2}(\alpha\sigma_y^{(2)}+\sigma_y^{(1)}+\sigma_y^{(3)})]=0$$

and this fixes $\alpha=-2$.

To order $1/m_N$ we obtained then for the anomalous magnetic-moment operator

$$(\mu_A)_y=\frac{2\sigma_y^{(1)}-(\sigma_y^{(2)}+\sigma_y^{(3)})}{2m_N}q^{(1)}+\frac{2\sigma_y^{(2)}-(\sigma_y^{(1)}+\sigma_y^{(3)})}{2m_N}q^{(2)}+\frac{2\sigma_y^{(3)}-(\sigma_y^{(1)}+\sigma_y^{(2)})}{2m_N}q^{(3)}.$$

Adding to this the Dirac moment, namely $(\sigma_y^{(1)}+\sigma_y^{(2)}+\sigma_y^{(3)})(q^{(1)}+q^{(2)}+q^{(3)})/2m_N$, we obtain for the total moments of the proton and the neutron

$$\mu_p^T=3e/2m_N,\quad \mu_n^T=-2e/2m_N,\tag{9}$$

which is in excellent agreement with the experimental result. Let us remark that the ratio $\mu_n/\mu_p=\frac{2}{3}$ follows immediately from assumption (C) but that the absolute value can only be fixed with the help of assumptions (A) and (B).

The physical meaning of the result becomes quite obvious if we write Eq. (9) as

$$\mu^T=\sum_{i=1}^{3}\frac{\sigma_y^{(i)}q^{(i)}}{2(m_N/3)}.$$

Because of their interaction, the quarks get an effective mass $\frac{1}{3}m_N$ and since the nucleon is in an

s state only the Dirac moments of the quarks contribute. Expressed in this way, the result is not surprising and has been known for a long time.[10]

An important problem is then to estimate the corrections, if any, to this result due to higher order terms in $1/m_N$. Looking at the angular condition, however, strongly suggests that Eq. (9) is exact (in the approximation we are working in, namely exact SU(3), pure s state, etc.). Indeed the higher orders in $1/m_N$ bring more and more powers of J_y in the expansion of the angular condition. The higher corrections to $\vec{x}$ and $\vec{x}'$ are then expected to be tensors[11] and therefore would not change our result. However, we have not been able to prove that an $M1$ contribution is actually excluded to all higher orders in $1/m_N$.

Finally, let us remark that in first order in $1/m_N$ the axial-vector coupling has, of course, the SU(6) value, namely $-G_A/G_V = 5/3$, but in this case the higher order terms do contribute.[3]

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[1]M. Gell-Mann, Phys. Rev. 125, 1067 (1962); Physics 1, 463 (1964).

[2]R. Dashen and M. Gell-Mann, Phys. Rev. Letters 17, 340 (1966).

[3]M. Gell-Mann, revised version of the lecture notes given at the 1966 International School of Physics "Ettore Majorana," Erice, 1966, edited by A. Zichichi (Academic Press, Inc., New York, 1966); California Institute of Technology Reports Nos. CALT-68-102 and CALT-68-103 (unpublished).

[4]We use the same notation as in Ref. 3. In particular, $\vec{k}$ will always be along the x direction.

[5]R. Dashen and M. Gell-Mann, in Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1966 (W. H. Freeman & Company, San Francisco, California, 1966).

[6]R. Gatto, L. Maiani, and G. Preparata, Phys. Rev. Letters 16, 377 (1966); H. Harari, Phys. Rev. Letters 16, 964 (1966); I. S. Gerstein and B. W. Lee, Phys. Rev. Letters 16, 1060 (1966).

[7]Besides the convergence problem, there is also a question of principle: The mass of the nucleon can be expressed in terms of the "external" variables ($M^2 = P_0{}^2 - \vec{P}^2$) but is also the eigenvalue of a given operator M which depends on the "internal" variables, i.e., $\vec{p}, \vec{p}', \vec{x}, \vec{x}'$. In the latter case, the explicit form of M is not known and, up to now, only the free quark-antiquark systems with a mass operator $M = 2(m^2 + p^2)^{1/2}$ (m is the quark mass and $\vec{p}$ the relative momentum) have been shown (Ref. 3) to be consistent with assumptions (A) and (B). Some further details are given in J. Weyers, Stanford Linear Accelerator Center Report No. SLAC-PUB-281 plus correctum (not to be published). Similar calculations have been made by M. Gell-Mann and D. Horn (private communication). The expansion in Eq. (7) then rests on the following assumption: There exists a mass operator M consistent with the algebra and the angular condition; furthermore, when sandwiched between nucleon states, this "internal" operator may be replaced by its "external" eigenvalue m_N!

[8]Since the nucleon is in an s state the orbital part is irrelevant for our purpose.

[9]As an operator $\Sigma_z{}^{(1)}$ does not satisfy the $\Delta J_x = 0, \pm 1$ condition but because of the s-wave assumption, its matrix elements do.

[10]W. Thirring, in Internationale Universitätswochen für Kernphysik der Karl-Franzens-Universität Graz. Fourth Schladming Winter School in Physics, 1965, edited by P. Urban (Springer-Verlag, Vienna, 1965). Gell-Mann (Ref. 3) obtained the same result but from a mass operator $M = 3m$ (m is the quark mass) and from an expansion of Eq. (7) in $1/m$ instead of $1/m_N$.

[11]As opposed to an axial current which would correct Eq. (9).