PROPAGATION OF ION-ACOUSTIC SOLITARY WAVES OF SMALL AMPLITUDE

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The conspicuous properties of the Kortweg-deVries equation have been investigated extensively by Zabusky and Kruskal.[1] In this note we show that the one-dimensional long-time asymptotic behavior of ion-acoustic waves of small but finite amplitude is described by the Kortweg-deVries equation in the same sense as was given by Gardner and Morikawa for a hydromagnetic wave in a cold plasma.[2] For a collisionless plasma of cold ions and warm electrons, the basic system of equations may be given as follows[3]:

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nu)=0, \quad (1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=E, \quad (2)$$

$$\frac{\partial n_e}{\partial x}=-n_eE, \quad (3)$$

$$\frac{\partial E}{\partial x}=n-n_e, \quad (4)$$

in which n and n_e denote the densities of ions and electrons, respectively, u is the flow velocity of ions, E the electric field, x the space coordinate, and t the time variable. All these quantities are dimensionless, being normalized in terms of the following characteristic quantities: a characteristic density n_0; the characteristic velocity $(\kappa T_e/M)^{1/2}$, where κ is the Boltzman constant, M is the ion mass, and T_e the electron temperature (T_e is assumed to be constant); the characteristic length, the Debye length $(\kappa T_e/4\pi e^2n_0)^{1/2}$; and the characteristic electric potential $\kappa T_e/e$.

Equations (1) and (2) are the continuity and the momentum equation for the cold-ion fluid, respectively; Eq. (3) is the momentum equation for the electron fluid in which the electron inertia is neglected; the last equation, Eq. (4), is the Poisson equation. We impose the boundary conditions that for $x\to\infty$,

$$n=n_e=1, \quad (5)$$

$$u=0. \quad (6)$$

Our aim is to derive the Kortweg-deVries equation, from the system of equations (1)-(4). The derivation goes parallel to that done by Gardner and Morikawa for the hydromagnetic wave in cold plasma propagating in a direction perpendicular to a magnetic field.

We first note that the dispersion relation of the linearized system leads to the phase

$$kx-\omega t=(x-t)\omega+\tfrac{1}{2}x\omega^3$$

for $\omega^2\ll 1$, where k and ω are the wave number and the frequency, respectively, or putting $\omega^2=\epsilon\mu^2$, we have the expression

$$kx-\omega t=\mu[\epsilon^{1/2}(x-t)+\tfrac{1}{2}\mu^2\epsilon^{3/2}x],$$

where ϵ is a small parameter. In view of this relation let us introduce the coordinates ξ and η through the equations[4]

$$\xi=\epsilon^{1/2}(x-t), \quad (7)$$

$$\eta=\epsilon^{3/2}x. \quad (8)$$

Then Eqs. (1)-(4) take the form

$$-\frac{\partial n}{\partial\xi}+\frac{\partial(nu)}{\partial\xi}+\epsilon\frac{\partial(nu)}{\partial\eta}=0, \quad (1')$$

$$-\frac{\partial u}{\partial\xi}+u\left(\frac{\partial u}{\partial\xi}+\epsilon\frac{\partial u}{\partial\eta}\right)=\tilde{E}, \quad (2')$$

$$\frac{\partial n_e}{\partial\xi}+\epsilon\frac{\partial n_e}{\partial\eta}=-n_e\tilde{E}, \quad (3')$$

$$\epsilon\frac{\partial\tilde{E}}{\partial\xi}+\epsilon^2\frac{\partial\tilde{E}}{\partial\eta}=n-n_e, \quad (4')$$

in which $\tilde{E}$ is defined by the equation

$$E=\epsilon^{1/2}\tilde{E}.$$

We now assume that n, n_e, u, and $\tilde{E}$ can be given in terms of the power series in ϵ, i.e.,

$$n=1+\epsilon n^{(1)}+\epsilon^2n^{(2)}+\cdots,$$

$$u=\epsilon u^{(1)}+\epsilon^2u^{(2)}+\cdots,$$

$$\tilde{E}=\epsilon\tilde{E}^{(1)}+\epsilon^2\tilde{E}^{(2)}+\cdots,$$

$$n_e=1+\epsilon n_e^{(1)}+\epsilon^2n^{(2)}+\cdots.$$

Then for the first and the second-order terms in ϵ, we have, respectively,

$$\frac{\partial n^{(1)}}{\partial\xi}=\frac{\partial u^{(1)}}{\partial\xi}=\frac{\partial n_e^{(1)}}{\partial\xi}=-\tilde{E}^{(1)}, \quad (9)$$

$$n^{(1)} = n_e^{(1)}, \tag{10}$$

and

$$\frac{\partial n^{(2)}}{\partial \xi} - \frac{\partial u^{(2)}}{\partial \xi} = \frac{\partial u^{(1)}}{\partial \eta} + \frac{\partial (n^{(1)} u^{(1)})}{\partial \xi}, \tag{11}$$

$$\frac{\partial u^{(2)}}{\partial \xi} + \tilde{E}^{(2)} = u^{(1)} \frac{\partial u^{(1)}}{\partial \xi}, \tag{12}$$

$$\frac{\partial n_e^{(2)}}{\partial \xi} + \tilde{E}^{(2)} = -\frac{\partial n_e^{(1)}}{\partial \eta} - n_e^{(1)} \tilde{E}^{(1)}, \tag{13}$$

$$n^{(2)} - n_e^{(2)} = \frac{\partial E^{(1)}}{\partial \xi}. \tag{14}$$

Under the boundary conditions, Eqs. (5) and (6), the first equality in Eq. (9) results in the identity

$$n^{(1)} = u^{(1)}. \tag{15}$$

Using these relations (9), (10), and (15) and eliminating the second-order quantities in Eqs. (11)-(14), we obtain the Kortweg-deVries equation,

$$\frac{\partial u^{(1)}}{\partial \eta} + u^{(1)} \frac{\partial u^{(1)}}{\partial \xi} + \frac{1}{2} \frac{\partial^3 u^{(1)}}{\partial \xi^3} = 0. \tag{16}$$

Also $n^{(1)}$, $n_e^{(1)}$, $u_e^{(1)}$, and $-\int \tilde{E}^{(1)} d\xi$ satisfy the same Eq. (16).

Since by means of Eqs. (7) and (8) $\eta = 0$ corresponds to $x = 0$, a solution of Eq. (16) satisfying an initial condition prescribed at $\eta = 0$, say $u^{(1)}(\xi, 0) = f(\xi)$, can be transformed to the solution in the (x, t) space satisfying the corresponding stretched boundary condition given at $x = 0$, i.e., $u^{(1)}(0, t) = f(-\epsilon^{1/2} t)$. If we employ, instead of Eq. (8), the transformation $\eta = \epsilon^{3/2} t$,[4] the initial value problems in the (ξ, η) space correspond to those in the (x, t) space. Thus, solving Eq. (16) numerically as well as analytically enables us to understand the asymptotic properties of the original system.

As an illustration, we derive from Eq. (16) a weak solitary wave of the original system, for which $u^{(1)}$ is a function of $\epsilon^{1/2}(x - u_0 t)$ only, namely,

$$u^{(1)} = g(u_0 \xi - a\eta), \tag{17}$$

where a is a constant and ϵa yields the difference between the wave velocity and the ion sound velocity, i.e., $u_0 = 1 + \epsilon a$. In the lowest order of ϵ, u_0 may be put equal to unity; then Eq. (17) becomes the usual form for the solitary wave in the (ξ, η) space. By virtue of Eq. (17), Eq. (16) takes the form

$$-ag' + u_0 g g' + \tfrac{1}{2} u_0^3 g''' = 0, \tag{18}$$

where the prime denotes the differentiation with respect to $u_0 \xi - a\eta$. If g, g', and g''' are set to be zero at infinity, integrating Eq. (18) twice, we have

$$(g')^2 = \frac{2a}{u_0^3} g^2 \left(1 - \frac{u_0}{3a} g\right), \tag{19}$$

and if and only if a is positive, i.e., the wave velocity exceeds the ion sound velocity, we obtain the compressive solitary wave

$$g = \frac{3a}{u_0} \operatorname{sech}^2\{(a/2u_0^3)^{1/2} \epsilon^{1/2} [(x - x_0) - u_0(t - t_0)]\}, \tag{20}$$

where x_0 and t_0 are arbitrary constants.

On the other hand, the exact solitary wave is given by the integral of the equation

$$\epsilon (u')^2 = 2(u - u_0)^{-2} \{\exp[-\tfrac{1}{2} u(u - 2u_0)] - u_0 u - 1\}, \tag{21}$$

which results immediately from Eqs. (1)-(4) under the assumption that all quantities depend on $\epsilon^{1/2}(x - u_0 t)$ only. It is easy to show that in the lowest order of the expansion in ϵ, Eq. (21) reduces to Eq. (19) when in the latter u_0 is replaced by unity.

From Eq. (20), it follows that the width of the solitary wave is of the order of $\epsilon^{-1/2}$ while its amplitude is of the order of ϵ. Therefore, the width of the solitary wave becomes larger for smaller amplitude. That is to say, steepening of the wave due to the weak nonlinearity is balanced by the dispersion in long wavelength so that the weak solitary wave is formed.

It should however be noted that Eq. (16) does not represent the totality of the solutions of the original system of Eqs. (1)-(4). For example, as can be seen easily from the above derivation of the solitary wave, Eq. (16) excludes the solitary wave propagating to the left. This is because of the assumption that ξ and η have the different dependences on the power of ϵ and the solution is obtained in the power of ϵ.

Finally, we explain briefly how Eq. (16) is related to the long-time asymptotic solution of the original system.[2] Linearizing Eqs. (1)-(4) leads to the following solution of the piston problem for large t:

$$\frac{u^{(1)}(x, t)}{U_1} \simeq \int_\beta^\infty \mathrm{Ai}(\alpha) d\alpha + \tfrac{1}{2}, \tag{22}$$

where $u^{(1)}(x, t)$ is small displacement, U_1 the piston speed directed to the right, $\mathrm{Ai}(\alpha)$ the Airy function, and

$$\beta = (\tfrac{2}{3})^{1/3}(x - t)/x^{1/3} = (\tfrac{2}{3})^{1/3} \xi / \eta^{1/3}.$$

On the other hand, linearizing Eq. (16) and assuming that $u^{(1)}$ depends on β only, we get the equation

$$u_{\beta\beta\beta}^{(1)}-\beta u_{\beta}^{(1)}=0,$$

which is satisfied by Eq. (22) and consequently describes the long-time asymptotic behavior of the original system.

[1]N. J. Zabusky and M. D. Kruskal, Phys. Rev. Letters 15, 240 (1965).
[2]C. S. Gardner and G. K. Morikawa, Courant Institute of Mathematical Sciences Report No. NYO 9082, 1960 (unpublished).
[3]B. D. Fried and R. W. Gould, Phys. Fluid 4, 139 (1961).
[4]The transformation is not unique; for example, the transformation $\xi=\epsilon^{1/2}(x-t)$, $\eta=\epsilon^{3/2}t$, which is implied by the relation $kx-\omega t=(x-t)k+\frac{1}{2}tk^3$ for $k^2\ll 1$, leads likewise to the Kortweg-deVries equation.

POTENTIOMETER FOR STUDYING THE LIQUID He II FILM*

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A simple level-sensing device is described for probing the profile of the chemical potential along the flowing liquid He II film. Measurements with these probes indicate that dissipation in the flowing film may occur either at highly localized regions or over extended regions of the flow path, depending upon experimental variations. Also it is shown that changes in transfer rate occurring during a given flow process are accompanied by a redistribution of the sites producing dissipation.

Studies of the flowing liquid He II film have a rather long history featured by numerous controversies over the observation and interpretation of "new" effects.[1] Unfortunately, a common ground for discussing these phenomena has not yet been developed. The present note describes a simple tool for investigating and analyzing the flow processes of the film.

During an informal conversation with Dr. B. D. Josephson concerning some unusual film-flow effects we had observed, he suggested the use of a "potentiometer" to probe the chemical potential μ_f along the path of the film.[2] The physical form that this probe might assume evolved in the discussion that followed.

Consider the flow of the film over the rim of a beaker with the liquid level z_i inside the beaker above the outside level z_0. The chemical potentials per atom of the two bulk liquids are, respectively, μ_i and μ_0; and their difference is given by

$$\mu_i-\mu_0=mg(z_i-z_0)=\Delta\mu. \quad (1)$$

On the basis of recent developments in the theory of superfluidity[3] we assume the following:

(1) $\Delta\mu$ provides the driving force for film flow.

(2) The bulk He II inside the beaker communicates with the bulk liquid outside through the film; at every point along the flow path S_f connecting the two reservoirs of bulk liquid the chemical potential is defined and is related to the phase of the order parameter through the relationship

$$\langle\psi\rangle=f\exp(i\varphi)=f\exp(i\mu t/\hbar).$$

For a film-flow experiment in which the path S_f is $z_i\to\text{rim}\to z_0$ and in which the distance along S_f is s, μ_f will have a profile $\mu_f=\mu_f(s)$.

(3) Production of dissipation in the form of quantized vorticity occurs through phase slippage only in those regions where $d\mu_f/ds$ is different from zero and at a frequency

$$\omega=d(\Delta\varphi)/dt=\Delta\mu/\hbar.$$

(4) In any region along S_f in which the velocity of particle flow $\vec{v}_s$ is equal to or exceeds the critical velocity $\vec{v}_{s,c}$, μ_f varies spatially; in regions where $\vec{v}_s<\vec{v}_{s,c}$, μ_f remains constant, provided $\vec{v}_s$ remains constant.

An example of the type of probe we have devised to investigate $\mu_f(s)$ is shown in Fig. 1. The side tube extending from the inside wall of the beaker has a radius small compared with that of the beaker itself, so that adjustments of the liquid level z_t in the tube by film flow occur rapidly compared with changes of z_i and z_0. We propose that subject to the as-