EFFECTIVE LONG-WAVELENGTH INTERACTION OF He^3 ATOMS DISSOLVED IN SUPERFLUID He^4

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He^3 atoms, when dissolved at low concentrations in superfluid He^4, have a weak residual interaction which is attractive at small momentum transfers. In a recent Letter Bardeen, Baym, and Pines[1] proposed a form for this interaction which yields Fermi-liquid parameters, for the He^3 in solution, in good quantitative agreement with those found experimentally by Anderson et al.[2] The effective interaction, assumed to be a velocity- and spin-independent potential $V(r)$, has a spatial Fourier transform given approximately by

$$V_k = V_0 \cos(\beta k/\hbar) \tag{1}$$

for k less than twice the Fermi momentum of a 5% solution. Explicitly $\beta = 3.16$ Å and

$$V_0 = -0.075 m_4 s^2/n_4, \tag{2}$$

where m_4, s, and n_4 are the mass, sound velocity, and density of pure He^4 at $T=0$.

The interaction (1) is surprisingly weak compared with the interatomic forces in the system, which are characterized by a strength $m_4 s^2/n_4$. As was pointed out in Ref. 1, the physical reason that the effective interaction is small is the following: In the solution the He^3 displaces the He^4; however, the force fields of He^3 atoms and He^4 atoms are identical and only the differences in zero-point motion and statistics of the He^3 atom from the He^4 background produce a residual interaction.[3] It is the purpose of this note to present a theoretical calculation, based on this idea, of V_0, the $k \to 0$ limit of the effective interaction.

Since $m_3^{-1} = m_4^{-1} + (3m_4)^{-1}$, where m_3 is the He^3 mass, the total Hamiltonian for the system of N_3 He^3 atoms dissolved in N_4 He^4 atoms can be written in the form

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_4} + \frac{1}{2}\sum_{i,j=1}^{N} U(r_i - r_j) + \sum_{j=1}^{N_3} \frac{p_j^2}{6m_4}, \tag{3}$$

where $N = N_3 + N_4$; the sum in the first term is over both the He^3 and He^4 particles, while the sum in the last term is only over the He^3. The sum in the middle term in (3) is over all $\frac{1}{2}N(N-1)$ pairs of particles. The important point here is that the interatomic potential U is the same for all pairs of particles, He^3-He^3, He^3-He^4, or He^4-He^4, in the system. Thus the first two terms in (3) are formally the Hamiltonian for N He^4 atoms; the last term is the kinetic-energy perturbation due to the He^3-He^4 mass difference.

In order to calculate V_0 we need the ground-state energy E of the system, and to evaluate this we take as a trial wave function the ground-state wave function $\Phi_0(N)$ of N He^4 atoms. This gives[4]

$$E = E_0(N) + N_3 E_1(N), \tag{4}$$

where $E_0(N)$ is the ground-state energy of N He^4 atoms, and

$$E_1(N) = \langle p^2/6m_4 \rangle \tag{5}$$

is one-third of the average kinetic energy per particle in a system of N He^4 atoms. A large part of the interactions of the He^3 with itself and the He^4 is accounted for in $E_0(N)$; the term N_3E_1 is essentially the extra zero-point energy of the He^3. Equation (4) is, in fact, an upper bound on the exact ground-state energy of the solution, and is exact only for $N_3 = 0$. Unfortunately this bounding property offers no indication as to the validity of the functional form of (4), linear in N_3 for fixed N; it is this functional form that is crucial to the result (9) for V_0. Physically this form derives from the fact that the last term in (3), the kinetic-energy perturbation, is a one-body operator, a sum of terms referring to the He^3 particles individually. A determination of the corrections to (4) requires a detailed microscopic calculation beyond the scope of this paper.

The long-wavelength limit of the effective He^3-He^3 interaction is given by the derivative of the chemical potential μ_3 for He^3 particles of one spin orientation with respect to a change at constant pressure of the density of He^3 particles of opposite spin (opposite in order to avoid exchange effects). The reason for constant pressure is that as the He^3 atoms move about, the He^4 atoms in their way are displaced in such a manner as to keep the local pressure constant. The fact that the trial function Φ_0 fails to take into account the exclusion principle is not a problem; for to determine V_0 at low concentrations, we need formally to consider a mixture containing at most two He^3 particles, of opposite spin orientation. For these two particles the exclusion principle does not apply, and (4) is a valid form for the energy.

From (4) the chemical potentials of the He^3 and He^4 are given by

$$\mu_3 = (\partial E/\partial N_3)_{N_4} = E_0' + E_1 + N_3 E_1',$$

$$\mu_4 = (\partial E/\partial N_4)_{N_3} = E_0' + N_3 E_1', \tag{6}$$

where the prime denotes the derivative with respect to N. We want then to compare μ_3 for $N_3 = 0$ and pressure P with its value for $N_3 = 1$ and the same pressure. To keep the pressure the same, one must change the volume Ω of the system slightly. Since in a large system the chemical potentials depend only on the particle density, one can equivalently readjust the total particle number N instead of the volume to keep the pressure constant.

The effective change in N is determined as follows: At constant temperature, $\Omega dP = N_3 d\mu_3 + N_4 d\mu_4$, so that in the limit of a few He^3 particles the condition of constant P is equivalent to constant μ_4. For $N_3 = 0$, $\mu_4 = E_0'(N_0)$, where $N_0 = n_4\Omega$. For $N_3 = 1$ we write $N = N_0 + \delta N$. Then the constancy of μ_4 implies that $E_0'(N_0 + \delta N) + E_1'(N_0 + \delta N) = E_0'(N_0)$; expanding in δN and neglecting terms of relative order $1/\Omega$, we have then

$$\delta N = -E_1'(N_0)/E_0''(N_0) \equiv -\alpha. \tag{7}$$

The change in μ_3 on adding one He^3 to pure He^4 is then

$$\begin{aligned}\delta\mu_3 &= E_0'(N_0 + \delta N) + E_1(N_0 + \delta N) \\ &\quad + E_1'(N_0 + \delta N) - E_0'(N_0) - E_1(N_0) \\ &= -\alpha^2 E_0''(N_0),\end{aligned} \tag{8}$$

neglecting terms of order $1/\Omega$ and using (7). Now in terms of the effective interaction, $\delta\mu_3 = V_0/\Omega$. Furthermore,

$$E_0''(N_0) = \partial\mu_4/\partial N_4 = m_4 s^2/n_4\Omega;$$

thus from (8),

$$V_0 = -\alpha^2 m_4 s^2/n_4. \tag{9}$$

The constant α can be evaluated in terms of the molar-volume data. Writing the molar volume of the solution in the form

$$v = v_{04}(P) + v_1(P)x, \tag{10}$$

where $x = N_3/(N_3 + N_4)$ is the concentration, we find simply that the change δN of the number of particles in volume Ω on adding one He^3 to pure He^4 is $\delta N = -v_1/v_{04}$, or

$$\alpha = v_1/v_{04}. \tag{11}$$

From the data of Kerr,[5] $\alpha = 0.28$, whereupon we have

$$V_0 = -0.078 m_4 s^2/n_4. \tag{12}$$

The agreement with the value $V_0 = -0.075 m_4 s^2/n_4$ derived empirically from fitting spin-diffusion data is well within that permitted by experimental uncertainty.

Lastly we estimate α theoretically. From Eqs. (5) and (7),

$$\alpha = \frac{n_4}{6 m_4 s^2}\frac{\partial\langle p^2\rangle}{\partial n_4}; \tag{13}$$

the right-hand side depends only on the properties of pure He^4. One can estimate the derivative in (13) without using detailed He^4 wave functions, by imagining each He^4 atom to be a hard sphere of diameter d moving independently in a spherical shell of radius $a - \frac{1}{2}d$ formed by the surrounding particles; a is some average interparticle spacing. The expectation value of p^2 in the ground state of the atom in the shell is then $\langle p^2\rangle = \pi^2\hbar^2(a-d)^{-2}$. Using this simple model de Boer[6] has found good agreement for the zero-point energy of solid He^4 by taking $a = 2^{1/6} n_4^{-1/3}$, appropriate to an fcc lattice, and the empirical value $d = 2.0$ Å; this is somewhat smaller than the hard-core diameter 2.6 Å appearing in the 6-12 potential. Assuming d to be independent of n_4 and $a \sim n_4^{-1/3}$, we have

$$\alpha = (\hbar/3 m_4 s a)^2 (1 - d/a)^{-3}. \tag{14}$$

Using de Boer's numerical value for d and $a = 3.8$ Å, an average "nearest-neighbor" distance, as determined by neutron scattering experiments,[7] we find from (14) that $\alpha = 0.28$; the agreement with the experimental value is surely coincidental.

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[3] See also I. Prigogine, The Molecular Theory of Solutions (North-Holland Publishing Company, Amsterdam, 1958).

[4] In the "ideal mixing" model of Prigogine and coworkers, the energy at $T = 0$ has the same linear dependence on N_3 as in Eq. (4); however, their coefficient of N_3 differs from the present one. The result (9) follows within their model as well. See I. Prigogine, Ref. 3 and I. Prigogine, R. Bingen, and A. Bellemans, Physica 20, 633 (1954).

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OSCILLATORY MAGNETOSTRICTION IN n-GALLIUM ANTIMONIDE

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We report here the first observation of oscillatory magnetostriction in a semiconductor, and show how these measurements, combined with data on the corresponding oscillations in the differential susceptibility, provide information about the effect of pressure on the band structure.

Both the longitudinal magnetostriction and the differential susceptibility were measured along the [110] direction in single crystals of gallium antimonide doped with tellurium. The magnetostriction was determined by means of a capacitance cell,[1] and the differential susceptibility was obtained by a field-modulation method.[2] The two sets of measurements were carried out to a maximum field of 35 kG at temperatures between 1.7 and 4.2°K.

Both the magnetostrictive strain ϵ and the differential susceptibility χ_d were found to be periodic functions of the reciprocal magnetic field $1/B$, with the same period, 2.5×10^{-6} G^{-1}, corresponding to quantum oscillations of a Fermi sphere containing 1.4×10^{18} carriers/cm^3. At a field of 35 kG at 4.2°K, the oscillation amplitude of ϵ was 7×10^{-9} and the oscillation amplitude of χ_d was 1.7×10^{-5} emu/cm^3. The period of these oscillations agrees very well with that of the Shubnikov–de Haas oscillations observed by Becker and Fan[3] in similarly doped n-GaSb, which is attributed to a spherical electron surface centered on the lowest conduction-band valley at $\vec{k} = (0, 0, 0)$. The temperature dependence of the magnetostriction oscillations gives an effective mass $0.055 m_0$, which agrees well with Becker and Fan's value $0.052 m_0$ for these carriers.

It is well known that the Landau quantization of the electronic energy levels in a magnetic field leads to quantum oscillations in a number of physical properties, including the ones mentioned above. In particular, oscillatory magnetostriction[4] has already been observed in bismuth,[5] beryllium,[6] and zinc.[7] We can obtain from the Lifshitz-Kosevich[8] expression for the free energy Ω of an electron gas in a magnetic field a Gibbs free energy G, which like Ω has a leading term periodic in $c\mathcal{A}/e\hbar B$, $\mathcal{A}$ being the extremal cross-sectional area of the Fermi surface giving rise to the quantum oscillations. The thermodynamic relations,

$$\epsilon_i = \left(\frac{\partial G}{\partial \sigma_i}\right)_H = -\left(\frac{c\mathcal{A}}{e\hbar B}\right) G' \frac{\partial \ln \mathcal{A}}{\partial \sigma_i},$$