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[1]S. Weinreb, M. L. Meeks, J. C. Carter, A. H. Barrett, and A. E. E. Rogers, Nature 208, 440 (1965); B. Zuckerman, A. E. Lilley, and H. Penfield, Nature 208, 441 (1965); A. H. Barrett and A. E. E. Rogers, Nature 210, 188 (1966); R. D. Davies, G. de Jager, and G. L. Vershuur, Nature 209, 974 (1966); D. D. Cudaback, R. B. Read, and G. W. Rougoor, Phys. Rev. Letters 17, 452 (1966); A. E. E. Rogers, J. M. Moran, P. P. Crowther, B. F. Burke, M. L. Meeks, J. A. Ball, and G. M. Hyde, Phys. Rev. Letters 17, 450 (1966).

[2]C. V. Heer and R. D. Graft, Phys. Rev. 140, A1088 (1965).

[3]H. De Lang and G. Bouwhuis, Phys. Letters 20, 383 (1966).

[4]G. C. Dousmanis, T. M. Sanders, Jr., and C. H. Townes, Phys. Rev. 100, 1735 (1955); H. E. Radford, Phys. Rev. Letters 13, 534 (1964).

[5]The notation of A. Messiah, Quantum Mechanics (North-Holland Publishing Company, Amsterdam, 1962), Vol. II, Appendix, is used.

MEANING AND RELIABILITY OF SOME THREE-BODY CALCULATIONS*

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Much of the work which is currently being done on the three-body problem is based on the Faddeev equations[1] in the separable approximation,[2,3] and on some simple heuristic extensions of the Faddeev equations to the relativistic case.[4-6] However, although many calculations of this type have been done, or are not undertaken, their limitations are far from being generally recognized. In fact, one cannot hope they will be able to describe thoroughly all the aspects of three-body interactions.

It is our purpose in this Letter to discuss the range of applicability of such models together with some of their important qualitative features.

(1) Practical limitations of the separable pole approximation.—In the nonrelativistic case, the two-body amplitude for the scattering of particles 2 and 3 depends upon the off-shell energy ζ_{23} of the two particles. This quantity, ζ_{23}, is related to the total energy z of the three particles by

$$\zeta_{23}=z-p_1^2/2m_1. \tag{1}$$

In the relativistic case, the relation between the total invariant energies squared σ_{23} and s is

$$\sigma_{23}=s-2\omega_1 s^{1/2}+m_1^2, \tag{2a}$$

where

$$\omega_1^2=p_1^2+m_1^2. \tag{2b}$$

Now the integration in the Faddeev equations is done from $p_1=0$ to infinity. If we keep to the relativistic case for definiteness, this implies that σ_{23} varies from $(s^{1/2}-m_1)^2$ to $-\infty$. The two-body amplitude is correctly represented by the separable-pole approximation to a resonance or bound state with mass m_B in the neighborhood of $\sigma_{23}=m_B^2$. Accordingly, if $s<(m_B+m_1)^2$, the whole integration in the Faddeev equations is made on a region where the two-body amplitude is unreliable.[7]

This is a very strong limitation of the approximation and, for instance, precludes any attempt to find the ω meson as a π-ρ bound state with such an approximation, as a recent paper nevertheless suggests.[6] (Let us note, furthermore, that in this particular example, the centrifugal factors displace the region of applicability of the equations much higher than $m_\rho+m_\pi$.) Many of the proposed applications suffer drastically from that criticism, which concerns not so much the separable aspect as the pole dominance.

(2) Influence of the Peierls singularities.—Above the particle + resonance threshold the pole approximation becomes reasonable. Also, it has been shown that it can generate three-body resonances which are remarkably stable under the approximations.[8-10]

Considering three-pion scattering in the $I=0$, $J^P=1^-$ channel, we can approximate the π-π scattering amplitude by keeping only the pole of the ρ meson.

In that case, the equations reduce to a one-variable integral equation with the kernel[9]

$$K(x,y)=-\frac{3\pi}{(2\pi)^3}\frac{(x^2-1)(y^2-1)^{1/2}}{D[\sigma(sy)]}\int_{-1}^{+1}\frac{d\zeta(1-\zeta^2)[x+y+a(\zeta)]}{a(\zeta)\{[x+y+a(\zeta)]^2-s\}}\frac{g[p(x,y,a(\zeta))]g[p(y,x,a(\zeta))]}{p(x,y,a(\zeta))p(y,x,a(\zeta))}, \tag{3}$$

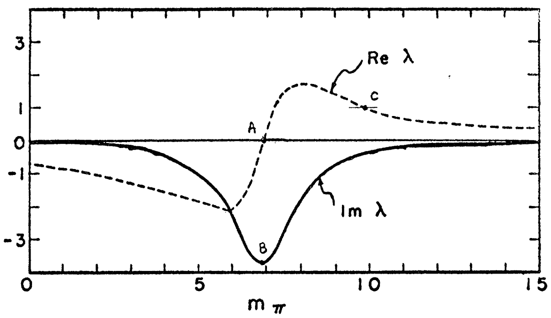

FIG. 1. Plot of the largest eigenvalue versus $\mathrm{Re}(s^{1/2})$ in pion masses for $\mathrm{Im}(s^{1/2}) = 0.4m_\pi$ (physical sheet). The dashed curve is the real part, the solid curve the imaginary part. Points A and B can characterize the position of the singularity. The three-body resonance will appear in the vicinity of point C, where the real part of the eigenvalue passes through 1, while its imaginary part is small.

where

$$a(\zeta) = [x^2 + y^2 - 1 + 2(x^2-1)^{1/2}(y^2-1)^{1/2}\zeta]^{1/2}, \quad (4a)$$

$$p(x,y,a) = \tfrac{1}{2}[(y+a)^2 - x^2 - 3]^{1/2}, \quad (4b)$$

and

$$\sigma(s,y) = s - 2ys^{1/2} + 1, \quad (4c)$$

and where s is the three-body invariant energy squared; we have taken $m_\pi = 1$. The function $g(p)$ is the ρ-π-π form factor, and $D(\sigma)$ is the two-body D function

$$D(\sigma) = c + \frac{1}{16\pi^2}\int_4^\infty \left(\frac{x-4}{x}\right)^{1/2} \frac{[g(\frac{1}{2}(x-4)^{1/2})]^2 dx}{x-\sigma}. \quad (5)$$

For a given functional form of the form factors, $D(\sigma)$, and therefore the constant c, is completely determined by the mass and width of the ρ.

Pinchings between the singularities of the Green's function and those of the function $1/D$ (branch point and pole) yield that as a function of s, the trace of the kernel has three singularities: the three-pion threshold $s = 9m_\pi^2$, the π-ρ threshold $s = (m_\rho + m_\pi)^2$, and the Peierls singularities at $s = 2m_\rho^2 + m_\pi^2$ and $s = [(m_\rho^2 - m_\pi^2)/m_\pi]^2$ of which only the first one, which is closest, may be important.[11] Any singularity of tr$K$ will also affect the eigenvalues.

Although it has been argued that the Peierls singularity is on a Riemann sheet which does not communicate with the physical sheet,[12] it is worth seeing numerically if it appears to affect the energy behavior of the eigenvalues in the physical region.

Preceding calculations[9] have shown that there is an eigenvalue λ of K, much larger than the others, which behaves as shown in Fig. 1. For the physical value of the ρ mass it has a singularity (visible as a bump in the imaginary part and a corresponding zero in the real part) which could not be directly ascribed to the π-ρ threshold or the Peierls singularity.[13]

We have let the ρ mass vary from $3m_\pi$ to $100m_\pi$ in order to follow the singularity. As visible from Table I, it follows quite closely the π-ρ threshold. Furthermore, no noticeable effect in λ or trace K could be detected near the Peierls singularity.

As a conclusion, even if the Peierls singularity may play a role in the properties of the three-body amplitude,[13] no connection, even

Table I. Positions of points A and B of Fig. 1 as the mass of the ρ is varied. M_ρ is the ρ mass; Γ_ρ is its width; $\mathrm{Im}(s^{1/2})$ is the distance above the real axis at which we operate; A and B are the positions of points A and B; $M_\rho + m_\pi$ is the abcissa of the π-ρ threshold; and P.S. is the abcissa of the closest Peierls singularity. All energies are expressed in pion masses. In the first three lines one can see that with the actual ρ mass and width one cannot conclude but that by decreasing Γ_ρ and $\mathrm{Im}(s^{1/2})$ we come closer to the singularity. It can be seen in the following lines to be quite close to the π-ρ threshold. We have checked that when $\Gamma_\rho \to 0$ and $\mathrm{Im}(s^{1/2}) \to 0$ the singularity remains, becomes more and more peaked, and points A and B come together at $M_\rho + m_\pi$.

M_ρ	Γ_ρ	$\mathrm{Im}(s^{1/2})$	A	B	$M_\rho + m_\pi$	P.S.
5.4	0.7	0.4	7.08	6.95	6.4	7.7
5.4	0.1	0.4	6.65	6.60	6.4	7.7
5.4	0.1	0.15	6.50	6.47	6.4	7.7
3.0	0.1	0.4	4.20	4.15	4.0	4.4
6.0	0.1	0.4	7.28	7.22	7.0	8.5
9.0	0.1	0.4	10.43	10.33	10.0	12.8
11.0	0.1	0.4	12.53	12.42	12.0	15.6
100.0	0.1	0.4	101.6	101.5	101.0	141.0

remote, can be established between the position of the singularity and the energy variation of the amplitude (including the position of resonances) in the equal-mass case. Let us notice that the relativistic aspects of the present calculation are not essential to the discussion.

(3) Spurious singularities at $s=0$.—We now turn to difficulties which appear specifically in relativistic Faddeev-type calculations.

As obvious in Eqs. (2a) and (4c), the "relativistic" Faddeev kernel that we have written in Eq. (3) has spurious singularities at $s=0$. These come from the $s^{1/2}$ factor which enters the equations in a nontrivial way.

Note that this does not appear directly in the Alessandrini-Omnes equations,[4] nor in those of Blankenbecler and Sugar,[5] but the reason for that, as these authors pointed out, was that the two-body amplitudes had an anomalous dependence on the energy of the third particle, and as a consequence the cluster property was not satisfied. Thus, in order to put the equations into a practical and understandable form, Basdevant and Kreps[8] replaced this anomalous dependence by Eq. (2a). Since the separable approximation is made from the beginning in the derivation of the equations of Freedman, Lovelace, and Namyslowski,[6] these authors obtain directly the equations used by Basdevant and Kreps.

Note also that this is a consequence of off-energy-shell relativistic kinematics, and will appear in any n-body calculation where one will want to reduce the Bethe-Salpeter equation to a linear integral equation involving three-momenta. Thus it is only in the two-body case, where a symmetric choice of variables is possible, that the application of the Blankenbecler-Sugar techniques[5] does not lead to such difficulties.

We will only sketch, here, the various aspects of these spurious singularities. The complete numerical investigation, which we have done, would be tedious, and will be given in a subsequent paper.

First there is a branch cut in the variable $E=s^{1/2}$. In fact, for negative values of E, we are not integrating over positive values of the two-body energy. Therefore, the two-body cut will reappear and give rise to a cut in the three-body energy (the eigenvalues are not real for $-3 \leq E < 0$). We have computed the discontinuity across this cut and it is quite important. Also, our calculations show that the pinching of the singularities mentioned in the second paragraph is now much more important than for positive values of E. A true "Peierls mechanism" seems to develop, and its discontinuity is very large. This can very strongly influence the magnitude of the eigenvalues at $s=0$.

But perhaps the most remarkable fact is that, for $s=0$, one can see from Eq. (2a) that only one two-body energy completely determines the three-body eigenvalues! In the calculation of tr$K$ mentioned above, the value of this quantity at $s=0$ does not decrease but tends to a constant value of the order of -1, when the ρ mass increases. Thus this singularity is important in determining the eigenvalues near $s=0$. The effect will be particularly marked in the case where two of the particles have a bound state at the mass of the third particle. In that case, at $s=0$ the D function vanishes, and we always integrate over an infinite function! One can easily see that in this situation an infinite number of bound state poles, with an accumulation at the origin, appear in each partial wave. This would happen for instance in a three-scalar-meson calculation if the scalar meson is itself considered as a bound state of two scalar mesons. This would also happen in a πNN calculation by putting in the nucleon as a bound state of πN (of course, in this latter case the region of interest is far from $s=0$).

A less drastic, but nevertheless serious, situation occurs when there is a virtual state near $\sigma=m_1^2$. Indeed, in that case, the on-the-energy-shell S function $e^{2i\delta}$ has a pole in the second Riemann sheet and a zero in the physical sheet where the scattering amplitude takes the value $-i/2q$, which is large. Accordingly, a calculation of the pion as a bound state of three pions, viz., the ABC virtual state, as done by Ahmadzadeh and Tjon,[14] is almost completely determined by the spurious singularity and is not reliable.

Our calculations have also shown that these spurious singularities do not strongly affect the qualitative features discussed in the second paragraph of this Letter. Therefore, these "relativistic" equations can still be used to investigate certain intermediate energy problems such as the properties of the A mesons.[10]

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[1]L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. 39, 1459 (1960) [translation: Soviet Phys.—JETP 12, 1014 (1961)].

[2]C. Lovelace, Phys. Rev. 135, B1225 (1964).

[3]J. L. Basdevant, Phys. Rev. 138, B892 (1965).

[4]V. A. Alessandrini and R. L. Omnes, Phys. Rev. 139, B167 (1965).

[5]R. Blankenbecler and R. Sugar, Phys. Rev. 142, 1051 (1966).

[6]D. Z. Freedman, C. Lovelace, and J. M. Namyslowski, Nuovo Cimento 43, 258 (1966).

[7]This trivial fact was first pointed out by J. L. Basdevant and R. E. Kreps [Phys. Rev. 141, 1398, 1404 (1966)] but its importance does not seem to have been fully realized elsewhere.

[8]Basdevant and Kreps, Ref. 7.

[9]Basdevant and Kreps, Ref. 7.

[10]J. L. Basdevant and R. E. Kreps, Phys. Rev. 141, 1409 (1966).

[11]R. E. Peierls, Phys. Rev. Letters 6, 641 (1961); 12, 50 (1964).

[12]R. Hwa, Phys. Rev. 130, 1580 (1963); C. Goebel, Phys. Rev. Letters 13, 143 (1964); G. Wanders, Helv. Phys. Acta 38, 142 (1965).

[13]All the eigenvalues, as well as tr$K$, have this general behavior in that region.

[14]A. Ahmadzadeh and J. Tjon, Phys. Rev. 139, B1085 (1965); 147, 1111 (1966).

PARTIAL CONSERVATION OF AXIAL-VECTOR CURRENT AND CHIRAL REPRESENTATION MIXING*

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It has been suggested by Dashen and Gell-Mann[1] that particle states form, in the limit $p_z \to \infty$, simple combinations of irreducible representations of the chiral $U(3)\otimes U(3)$ group. Recently, several authors[2-5] considered this problem and treated explicit examples of such representation-mixing schemes for baryonic states. The properties of the particles that were calculated from theory and compared with experiment were all either weak or electromagnetic ones; namely, axial-vector couplings (g_A and G^*) and their D/F ratio and ratios of magnetic moments.

In the present Letter we use partial conservation of axial-vector current (PCAC) to evaluate the axial-transition amplitudes from known π coupling constants. This way we can analyze the data obtained from strong interactions and compare them with the group theoretical predictions. This will be applied, as a test, to mixing schemes for baryons, and it will enable us to deal with mesons, too.

Let us define A^i as the axial "charge"

$$A^i = \int d^3x\, j_0^{Ai}, \tag{1}$$

where i is an isospin index. Using PCAC we can write the matrix element of A^i between different mass states as[6]

$$\begin{aligned}\langle Ap_1|A^i|Bp_2\rangle &= -\int d^4x\,\theta(x_0)\langle Ap_1|\partial_\mu j_\mu^{Ai}|Bp_2\rangle\\ &= -C(2\pi)^3\delta^3(\vec{p}_1-\vec{p}_2)\int\frac{dx_0\,\theta(x_0)}{(E_1-E_2)^2-m_\pi^2}\exp[i(E_1-E_2)x_0]\langle Ap_1|j_\pi^i(0)|Bp_2\rangle,\end{aligned} \tag{2}$$

where

$$\partial_\mu j_\mu^{Ai} = C\varphi_\pi^i = \frac{m_N g_A m_\pi^2}{g_{NN\pi}K_{NN\pi}(0)}\varphi_\pi^i. \tag{3}$$

In the following we will assume that $K_{NN\pi}(0)$, as well as all other π-vertex form factors evaluated at $q^2=0$, is approximately 1. [The assumption that is necessary for our purposes is $K_{AB\pi}(0)=K_{NN\pi}(0)$.]

Now let us define

$$\lim_{p_z\to\infty}|\langle Ap_1|A^i|Bp_2\rangle| = a^i(A,B)\delta^3(\vec{p}_1-\vec{p}_2), \tag{4}$$

and the relativistically invariant amplitudes

$$(2\pi)^3(4E_1E_2)^{1/2}\langle Ap_1|j_\pi^i(0)|Bp_2\rangle = F_{AB}^i(p_1p_2) \quad \text{for bosons,} \tag{5a}$$