[6]C. L. Chen, Phys. Rev. 135, A627 (1964); G. G. Comisar, Phys. Fluids 6, 76 (1963).

[7]D. Bohm and E. P. Gross, Phys. Rev. 79, 992 (1950).

[8]National Applied Mathematics Laboratory, National Bureau of Standards, Tables Relating to Mathieu Functions (Columbia University Press, New York, 1951).

STABILITY OF SUPERFLUID FLOW IN AN ANNULUS*

R. J. Donnelly
Department of Physics and Institute for the Study of Metals, The University of Chicago, Chicago, Illinois

and

Alexander L. Fetter
Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California
(Received 13 July 1966)

The flow of superfluid in an annulus is considered on the basis of free-energy arguments. Two critical velocities appear: One is equivalent to comparing the free energy of the fluid with the free energy of solid-body rotation; the other is equivalent to making the free energy of the fluid zero and corresponds to Feynman's criterion.

The behavior of rotating He II has recently been studied in detail with superfluid gyroscopes.[1,2] A persistent current in a complicated multiply connected geometry is generated either by rotating the sample above T_λ and then cooling, or by rotating below T_λ. In both methods, the container is subsequently brought to rest, and the angular momentum of the persistent current is detected gyroscopically. The magnitude of the final persistent current depends both on the initial angular velocity and on the method of production. In particular, several critical-velocity effects have been observed.[3] The present note contains a theoretical discussion of superfluid flow in a multiply connected region. Although this work is restricted to a two-dimensional annulus ($R_1 < r < R_2$), the conclusions are insensitive to the detailed shape of the container and should also be applicable to more complex experimental configurations. The details of these calculations will be published separately.

The low-lying states of an inviscid fluid in a rotating container are conveniently characterized by the free energy $F = E - \Omega L$ where E and L are the total energy and angular momentum.[4] The equilibrium state at a given angular velocity Ω is taken to be that which minimizes the free energy. Quantum mechanics affects the theory only through the quantization of circulation in units of $\kappa = h/m$, where h is Planck's constant and m is the mass of a helium atom. This approach is valid for a general annulus ($0 < R_1/R_2 < 1$); in the special case of a wide annulus ($R_1 \ll R_2$), our expressions reproduce Vinen's[5] results for fluid in a cylinder containing a fine wire. We consider here only the more interesting limit of a narrow annulus, where the width $d \equiv R_2 - R_1$ is much smaller than the mean radius $R \equiv \frac{1}{2}(R_2 + R_1)$. At low angular velocities the equilibrium flow is a purely irrotational circulation with tangential velocity $v(r) = \Gamma/2\pi r$.[6] The equilibrium circulation Γ is uniquely determined by the applied angular velocity, and the nth quantum state ($\Gamma = n\kappa$) is energetically favorable only in a narrow range

$$(n - \tfrac{1}{2})\kappa/2\pi R^2 < \Omega < (n + \tfrac{1}{2})\kappa/2\pi R^2. \quad (1)$$

The quantized circulation states form a sequence of equally spaced levels and represent the equilibrium configuration up to a critical angular velocity

$$\Omega_0 = (\kappa/\pi d^2)\ln(2d/\pi a), \quad (2)$$

at which point singly quantized vortices (with core radius a) appear in the bulk of the fluid. The maximum circulation for irrotational vortex-free flow is

$$\Gamma_{max} = 2\Omega_0 \pi R^2 = 2\kappa(R/d)^2 \ln(2d/\pi a), \quad (3)$$

which is much larger than κ.[7] For $\Omega \gtrsim \Omega_0$, the vortices are equally spaced on the circumference of a circle midway between the walls, and their number increases rapidly with Ω. The magnitude of the circulation is essentially unaffected by the appearance of the vortices, which rotate with the external angular velocity Ω of the container. This behavior is necessary for self-consistency (i.e., the vortices must move with the normal fluid at the velocity of the container) and occurs here precisely because of the large circulation about the inner cylinder. If $\Omega \gg \Omega_0$, the fluid is uniformly and densely filled with vortices at a density $2\Omega/\kappa$; the equilibrium circulation is given by $\Gamma_{eq} = 2\Omega\pi R^2$.[4]

A notable feature of Eq. (2) is that Ω_0 is independent of R, which can be understood qualitatively by the following observation: The equilibrium circulation about the inner cylinder ($\Gamma_{eq} = 2\Omega\pi R^2$) is equal to the value that would occur if the inner cylinder were uniformly filled with singly quantized image vortices at a density $2\Omega/\kappa$. Physical vortices are not expected to appear in the fluid until the mean spacing of the image vortices is comparable with the width of the annulus, i.e., $(\kappa/2\Omega_0)^{1/2} \approx d$, in approximate agreement with Eq. (2).

The treatment just given assumes that the fluid actually attains the state of lowest free energy. Unfortunately, there may be experiments with He II for which this assumption is not always realistic, and we have consequently considered the effect of constraining Γ to vanish identically. With this additional restriction, the fluid remains stationary until it becomes possible to create vortices at a distance Ca from the inner wall, where C is a constant of order unity.[8] The critical angular velocity for this process is given by

$$\Omega_c = (\kappa/4\pi dR)\ln(2C), \tag{4}$$

which is smaller than Ω_0 by a factor d/R. A similar calculation for creation of vortex pairs yields a critical angular velocity approximately twice as large. Although a purely irrotational state with circulation $\Gamma = 2\Omega_c\pi R^2$ has a lower free energy than any configuration of vortices at this angular velocity, such a velocity pattern is prohibited by the assumed constraint on the circulation.

Our calculations have led to two distinct critical angular velocities each of which has an associated critical linear velocity:

$$v_0 = (\kappa R/\pi d^2)\ln(2d/\pi a),$$
$$v_c = (\kappa/4\pi d)\ln(2C). \tag{5}$$

The larger velocity v_0 becomes infinite with R/d; the smaller one v_c is independent of R and is just the Feynman critical velocity[9,10] for the onset of vortex production in a channel of width d. With typical geometries the two critical velocities can differ by several orders of magnitude: For $R = 1$ cm, $a = 10^{-8}$ cm, $d = 10^{-4}$ cm, we find $v_0 \approx 3\times10^5$ cm sec^{-1}, $v_c \approx 1$ cm sec^{-1}; while a wider gap $d \approx 0.1$ cm leads to $v_0 \approx \frac{1}{2}$ cm sec^{-1}, $v_c \approx 10^{-3}$ cm sec^{-1}.

These critical velocities arise in distinct physical contexts, and it is not surprising that they can differ so greatly. The calculation of v_0 is approximately equivalent to minimizing the relative velocity between the rotating walls and the moving fluid subject to the condition of quantized circulation. Thus it becomes favorable to change Γ by one unit whenever $|\Gamma/2\pi R - \Omega R|$ is the order of $\kappa/2\pi R$. Similarly, physical vortices appear at an angular velocity Ω_0, when a single vortex can just compensate for the difference in irrotational velocity between the inner and outer walls,

$$\Omega_0 R \approx \Gamma/2\pi(R+\tfrac{1}{2}d) + (\kappa/\pi d)$$
$$\approx \Gamma/2\pi(R-\tfrac{1}{2}d) - (\kappa/\pi d). \tag{6}$$

This predicts a maximum irrotational circulation of order $\kappa(R/d)^2$, which agrees with Eq. (3) apart from the logarithmic factor. In contrast, the Feynman criterion answers the very different physical question: What is the largest relative fluid velocity that can be sustained between the superfluid and the walls? In the annular channel the superfluid remains at rest until the container reaches the angular velocity Ω_c. At Ω_c, vortices form near the inner wall and the number of vortices increases until the fluid is moving at the angular velocity of the container. The calculation of v_c is equivalent to making the free energy of the vortex array equal to the free energy of the fluid at rest, $F = 0$. In a straight channel, the superflow (supplied by some external source) may be increased until vortices appear at velocity v_c; a thermodynamic potential then develops across the channel, and further increase in velocity is prevented.[10]

The unrestricted free-energy calculations are directly applicable to the experimental situation of rotation above T_λ followed by cooling. For $\Omega < \Omega_0$, the fluid remains vortex free and has an angular momentum

$$L_{eq} = \rho_s \Gamma_{eq} Rd \approx 2\pi\rho_s \Omega R^3 d, \tag{7}$$

which is just the value for solid-body rotation. Equation (7) agrees with the observations made at low angular velocities by Mehl and Zimmerman,[3] who, for practical reasons, stop their container before measuring the persistent currents. Hence, the magnitude of the final persistent current is, in fact, limited not by Ω_0 but by the Feynman critical velocity v_c. They find that the linear relation between L_{eq} and Ω breaks down above 2.5 rad sec^{-1}, which is approximately the Feynman value expected for their extremely fine channels ($d \approx 10^{-4}$ cm).

The restricted free-energy calculations are applicable when rotation is started well below T_λ. With this procedure, Mehl and Zimmerman[3] find no persistent current for $\Omega \lesssim 2.5$ rad sec^{-1}, above which a finite current is observed. This result is consistent with the theory outlined above.

In principle it is also possible for the fluid to pass reversibly into irrotational flow as the container accelerates from rest; the fluid would then return to rest when the container is stopped. This behavior is probably prevented by the requirement that a phase-coherent disturbance must be present which changes the velocity of the superfluid throughout the entire annulus. At present there is no experimental evidence that the circulation Γ can change reversibly.[11]

It is interesting to consider whether the variour stable states discussed here can be understood in terms of the de Broglie wavelength $\lambda = h/mv$ associated with superfluid flow.[10] Since λ characterizes the distance through which the flow has "quantum coherence," a velocity pattern with a characteristic length l cannot occur at an arbitrarily low velocity but instead occurs only when λ is comparable with l. States of pure quantized circulation ($\Gamma = n\kappa$) follow directly from the condition that n de Broglie wavelengths fit into circumference of the annulus:

$$2\pi R = n\lambda = nh/mv = n\kappa/\Omega R, \tag{8}$$

suggesting that the nth state is stable at an angular velocity $\Omega \approx n\kappa/2\pi R^2$, which agrees with Eq. (1). On the other hand, the application of this argument to vortex creation suggests that the critical angular velocity occurs when $\lambda \approx d$, leading to the Feynman value

$$\Omega_c = O(\hbar/mdR). \tag{9}$$

The de Broglie wavelength criterion is incapable of predicting Ω_0 because Eq. (2) requires a detailed analysis of combined states containing both circulation and vortices.

It must be pointed out that free-energy considerations may not be sufficient to characterize the onset of instability for irrotational flow. Examples from classical hydrodynamics[12] show that perturbation methods are more reliable since they do not assume a particular form for the final stable state. The de Broglie wavelength criterion represents a crude first attempt at such a calculation, but a detailed perturbation analysis of the onset of instability in quantum hydrodynamics would be extremely valuable.

An experimental search for vortices at Ω_0 would decide the relevance of these free-energy calculations. This might be done through attenuation of second sound, or through ion trapping. A wider annulus ($d \approx 0.1$ cm) would simplify such measurements, which must be performed while the apparatus is in rotation. The importance of such an experiment should compensate for its difficulty.

We are grateful for discussions with Professor H. E. Hall, Professor W. A. Little, Dr. J. B. Mehl, and Professor W. Zimmerman.

---

*The research of R.J.D. is supported by the National Science Foundation under Grant No. GP-5302 and by the U. S. Air Force Office of Scientific Research under Grant No. AF-AFOSR 785-65. Facilities provided by Advanced Research Projects Agency for materials research at the University of Chicago have also been of benefit. The research of A.L.F. is supported in part by the U. S. Air Force through Air Force Office of Scientific Research Contract No. AF 49(638)-1389.

[1]J. D. Reppy, Phys. Rev. Letters 14, 733 (1965).

[2]J. B. Mehl and W. Zimmerman, Jr., Phys. Rev. Letters 14, 815 (1965).

[3]J. B. Mehl and W. Zimmerman, Jr., Bull. Am. Phys. Soc. 11, 479 (1966); and private communication.

[4]See, for example, A. L. Fetter, to be published.

[5]W. F. Vinen, Proc. Roy. Soc. (London) A260, 218 (1961).

[6]P. J. Bendt and T. A. Oliphant, Phys. Rev. Letters 6, 213 (1961); P. J. Bendt, Phys. Rev. 127, 1441 (1962).

[7]It appears that M. P. Kemoklidze and I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. 46, 1677 (1964) [translation: Soviet Phys.—JETP 19, 1134 (1964)], would predict a similar value although they do not give an explicit expression for Ω_0 or Γ_{max}.

[8]C may be considered as a correction for quantum effects at the core; see, for example, A. L. Fetter, Phys. Rev. Letters 10, 507 (1963).

[9]R. P. Feynman, in Progress in Low-Temperature Physics, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1955), Vol. I, p. 17.

[10]R. J. Donnelly, Phys. Rev. Letters 14, 939 (1965); also R. J. Donnelly, in Non Equilibrium Thermodynamics, Variational Techniques and Stability, edited by

R. J. Donnelly, R. Herman, and I. Prigogine (University of Chicago Press, Chicago, Illinois, 1966), p. 273.

[11]Such behavior may occur near T_λ. A similar effect occurs in a superconducting cylinder where reversible changes of quantized flux are observed near T_c but cannot take place at lower temperatures. See R. D. Parks and W. A. Little, Phys. Rev. 133, A97 (1964).

[12]S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, (Clarendon Press, Oxford, England, 1961); also C. C. Lin, The Theory of Hydrodynamic Stability (Cambridge University Press, Cambridge, England, 1955).

LOW-TEMPERATURE PROPERTIES OF NEARLY FERROMAGNETIC FERMI LIQUIDS

S. Doniach and S. Engelsberg*†
Physics Department, Imperial College, London, England
(Received 15 August 1966)

Effective-mass corrections due to emission and reabsorption of persistent spin fluctuations are shown to be large in Pd and to have a temperature dependence which fits qualitatively that of the low-temperature specific heat of He^3.

A characteristic feature of two strongly interacting Fermi liquids, namely the d-band holes in metallic palladium, and liquid He^3, is the considerable enhancement of the observed static paramagnetic susceptibility at low temperatures over the Pauli susceptibility deduced from the observed value of the low-temperature specific heat using noninteracting Fermi-gas theory. The work of Slater and of Stoner[1] on the magnetic properties of fermion systems shows that the physical origin of this enhancement can be represented, within the framework of a molecular field or random phase approximation (RPA), as resulting directly from a semiphenomenological short-range repulsion between fermions. We take this to be

$$H_{\text{int}} = I\int d^3x\, n_\uparrow(\vec{x})n_\downarrow(\vec{x}')\delta(\vec{x}-\vec{x}'), \tag{1}$$

where $n_\uparrow(\vec{x}) = \psi_\uparrow^\dagger(\vec{x})\psi_\uparrow(\vec{x})$ is the spin-up number density.

In this Letter we show that there is a class of corrections to RPA which become increasingly important as the exchange enhancement is increased, and we discuss some of their consequences. These corrections are the contributions to the one-particle self-energy, and hence effective mass, arising from the emission and reabsorption of persistent spin fluctuations (or critically damped spin waves) analagous to the phonon corrections to the effective mass resulting from electron-phonon interaction.[2] These are taken into account by summation of the particle-hole t-matrix contribution to the self-energy. The propagator for the spin fluctuations, which results from this summation, is defined by

$$\chi^{-+}(\vec{x}-\vec{x}', t-t') = +i\langle T\{\sigma^-(\vec{x},t)\sigma^+(\vec{x}',t')\}\rangle, \tag{2}$$

where $\sigma^+(\vec{x}) = \psi_\uparrow^\dagger(\vec{x})\psi_\downarrow(\vec{x})$. This has been discussed by Izuyama, Kim, and Kubo[3] who show that the Fourier transform of (2), $\chi(\vec{k},k_0)$, is given in RPA in terms of $\chi^0(\vec{k},k_0)$ for the noninteracting Fermi gas by

$$\chi^{-+}(\vec{k},k_0) = \frac{\chi^0(\vec{k},k_0)}{1-I\chi^0(\vec{k},k_0)}. \tag{3}$$

The persistence of the spin fluctuations represented by χ is now seen by looking at χ in the region of small $|\vec{k}|/p_F$ when χ^0 may be expanded in powers of k/p_F to give

$$\chi^{-+}(\vec{k},k_0) = \frac{N(0)}{K_0^2 + (\bar{I}/12)\bar{k}^2 - i(\pi/4)\bar{I}|k_0|/\bar{k}} \tag{4}$$

for $\bar{k}_0 = k_0/\epsilon_F < 2\bar{k} = 2|\vec{k}|/p_F \ll 1$. For $\bar{k}_0 \gtrsim 2\bar{k}$ the imaginary term in the denominator goes to zero. $N(0)$ is the density of states, $mp_F/2\pi^2$, at the Fermi level; $\bar{I} = IN(0)$; and $K_0^2 = (1-\bar{I})$ is the inverse of the Stoner enhancement factor. For paramagnets for which $K_0^2 \ll 1$, this function is strongly peaked at $k_0 \cong \epsilon_F K_0^2 \bar{k}$ so that although the spin fluctuations are strongly damped, their characteristic excitation energy is reduced by the factor K_0^2 by the effect of the interactions, i.e., they tend to persist for very long times as $K_0^2 \to 0$ at which point the Stoner criterion for transition to the ferromagnetic state is satisfied.

We now consider the following correction to the ground-state energy based on the use of the RPA formula (3) for the particle-hole correlation to evaluate $\langle H_{\text{int}}\rangle$, which may then be integrated with respect to the coupling constant