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MAXIMUM LOSSLESS CURRENT IN A SUPERCONDUCTING FOIL WITH A SURFACE SHEATH*

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The maximum lossless transport current density of the superconducting surface sheath of a long foil of rectangular cross section is controlled by the larger dimension $2a$ of the cross section when the applied magnetic field is parallel to $2a$ and perpendicular to the current. When $2a$ becomes very large the maximum current density becomes very small.

The maximum current density of the superconducting surface sheath has attracted a great deal of theoretical attention recently.[1-4] In particular, it was predicted from energy considerations[3] that the magnetization per unit volume of a long cylinder due to persistent currents which flow around the axis of a cylinder should be size dependent. This is in quantitative agreement with recent experiments on good surfaces.[5,6] It was also predicted that the maximum current density of a surface sheath which is infinite in two dimensions is finite,[1,2] and that for a foil which is infinite in two dimensions with one or both surfaces superconducting the maximum current density depends[4] on the thickness $2b$ of the foil.

We shall show that for an infinite surface sheath or a foil in two dimensions the maximum lossless current density is zero, contrary to the earlier prediction.[1,2,4] However, a foil of thickness $2b$, of width $2a$, and of infinite length can carry a finite lossless surface current density whose magnitude is controlled by the width $2a$ and not by the thickness $2b$ when $(b/a)^2 \ll 1$ and $b > \Delta$ (Δ is the thickness of the surface sheath).

We employ the same physical principles as in Ref. 3, namely, that the Gibbs free-energy difference $\Delta G_{SN}(H_0)$ between the superconducting state in a magnetic field with a current and the normal state (assumed nonmagnetic) without a current is zero for maximum lossless current in the surface sheath. Park[4] states that he uses the same criterion whereas, in fact, he equates the free energy of the superconducting and normal states with a current flowing in each. $\Delta G_{SN}(H_0)$ may be written[3]

$$\oint dV\{(\vec{H}-\vec{H}_0)^2-\tfrac{1}{2}|\Psi|^4\}=\Delta G_{SN}(H_0), \qquad (1)$$

where the order parameter $\Psi(x,y,z)$ and the magnetic field $\vec{H}(x,y,z)=\operatorname{curl}\vec{A}(x,y,z)$ have to be determined from the Ginzburg-Landau[7] equations and Eq. (1). $\vec{H}(x,y,z)$ is the local magnetic field at the applied magnetic field H_0. Equation (1) is written in the usual Ginzburg-Landau normalization[7] and the integral is to be extended over all space. When one neglects the more sophisticated details of the internal current distribution of the surface sheath, Eq. (1) means roughly the following: The first term on the left-hand side is the total energy which arises from a total current I in the specimen ($J=I/2a$). The consequence of this current at the applied field H_0 is the magnetic field $(H-H_0)$

$=H_J$. The maximum current J_c is then determined from the condition that the total magnetic energy from the current is balanced by the total available configurational energy of the superconductor which is the second term on the left-hand side of Eq. (1). When a persistent circulating current flows around the axis of a very long cylinder,[3] the magnetic field H_J is stored entirely inside the specimen and, therefore, it is sufficient to integrate Eq. (1) over the specimen volume. However, quite the opposite is true when one passes a current I along the surface of a rectangular foil whose cross section is of area $4ab$ when the applied magnetic field is parallel to the larger dimension $2a$ of the cross section and perpendicular to I. If we assume that equal lossless currents flow on both surfaces of the foil and no current flows in the resistive bulk of the specimen, the stored magnetic energy due to J is essentially zero over the specimen volume except near the surface when we have, for example, $b \gg \Delta$. The magnetic field H_J is essentially stored outside the specimen and, therefore, one has to integrate Eq. (1) over all space and not just over the specimen volume.

The order parameter Ψ at a given magnetic field and a given Ginzburg-Landau κ value is not a very sensitive function of the critical current, and one may to the first approximation choose the value of Ψ corresponding to the lowest energy ($J=0$) for given values of H_0 and κ. This has been justified theoretically[3,8] and agrees quantitatively with experiments.[5,6] When the sheath is in its lowest energy state, the maximum internal current density is not zero (though the total current is zero) but large compared to the critical current density. The critical current is only a perturbation[3] on the inherent currents and thus does not influence $\Psi(x,y,z)$ very strongly for specimens whose dimensions are large compared to the coherence length ξ.

Assume that the center of the coordinate system is in the center of the foil which covers the region $-a \leq x \leq a$ and $-b \leq y \leq b$. The current I flows parallel to the positive z direction in which direction the sample is assumed to extend to infinity. The uniform magnetic field H_0 is parallel to the positive x direction and with $\Delta G_{SN}(H_0)=0$, Eq. (1) reduces to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_J^{\,2}(x,y)\,dy\,dx = \tfrac{1}{2}\int_{-a}^{+a}\int_{-b}^{+b}\Psi^4(y)\,dy\,dx. \qquad (2)$$

Ψ is assumed to be independent of x if we assume that the self-field effects at the ends of the specimen near $\pm a$ can be neglected ($H_J \ll H_0$). Outside the specimen, $\Psi=0$. With the following definitions:

$$\Delta = \frac{1}{|\Psi(b)|^2}\int_0^b \Psi^2(y)\,dy, \qquad (3)$$

$$\beta = \int_0^b \Psi^4(y)\,dy\,[\int_0^b \Psi^2(y)\,dy]^2, \qquad (4)$$

the right-hand side of Eq. (2) is $2a\beta\Delta^2 F^4(b)$ where β is a parameter of order unity. To solve the double integral on the left-hand side of Eq. (2) we imagine that a current I_2 flows in the positive z direction and a current I_1 in the negative z direction. I_2 flows over the total cross section of the foil, and I_1 flows only over $-(b-\Delta) \leq y \leq (b-\Delta)$ and $-a \leq x \leq a$. The current densities are assumed to be equal over the latter cross section but of opposite sign such that the total current density is zero in the resistive core of the foil. The measured current $I=I_2-I_1$ flows then near the surfaces and the measured current density is $j=I/4a\Delta$. The vector potential associated with I_2 is A_2 and that with I_1 is A_1. By partial integration the left-hand side of Eq. (2) reduces to

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(A_1+A_2)\nabla^2(A_1+A_2)\,dx\,dy = \frac{8\pi}{c}j\int_{b-\Delta}^{b}\int_{-a}^{+a}(A_1+A_2)\,dx\,dy, \qquad (5)$$

because A_1 and A_2 and the corresponding magnetic fields become zero at $x=y=0$ and at $x=\infty$ and $y=\infty$, and the total transport current density is finite only over the surface layer Δ over which we assumed that j is a constant[9] (c is the velocity of light). The vector potential for a long rectangular foil can be calculated directly or obtained from Strutt,[10] and when terms of order $(b/a)^2$ and smaller are neglected with respect to unity, one obtains

$$A_{1,2} = \mp\frac{I_{1,2}}{c\,2a}\left[(a+x)\ln\frac{(a+x)^2+y^2}{a^2}+(a-x)\ln\frac{(a-x)^2+y^2}{a^2} + 2y\left(\arctan\frac{a+x}{y}+\arctan\frac{a-x}{y}\right)\right]. \qquad (6)$$

When Eq. (6) is substituted into the right-hand side of Eq. (5) and integrated, then it follows from Eq. (2) that the maximum or critical transport current in real units is

$$\frac{4\pi}{c}J_c = 2\left(\frac{\beta\pi}{\ln 4-1}\right)^{1/2}\left(\frac{\lambda}{a}\right)^{1/2} H_c\,\frac{\Delta}{\xi}\,\frac{F^2(b)}{\kappa}, \qquad (7)$$

where $J_c = I_c/2a$ and Eq. (7) is written in cgs Gaussian units. The functions $(\Delta/\xi)\ [F^2(b)/\kappa]$ are tabulated together or in part in Refs. 3 and 5 and by Fink.[11] When a superconducting sheath exists only on one side of the foil the measured critical current is by a factor of $\sqrt{2}$ smaller than that given by the right-hand side of Eq. (7). Equation (7) is quite different from what Park[4] suggests, and it shows that the critical current density of a surface sheath which extends to infinity in two dimensions is zero, contrary to the conclusions of Abrikosov[1] and Park.[2]

If one deals with a semi-infinite superconducting half-space it follows from Maxwell's equations that the total current (per unit length) which flows in the specimen is proportional to the difference of the magnetic field at the boundary surface $H(b)$ and at infinity $H(\infty)$ [in the bulk]: $(4\pi/c)J = H(b) - H(\infty)$. Park,[2] however, indicates that he deliberately kept the magnetic field uniformly constant over all space so that his calculated critical currents should be zero even when one disregards the energy consideration.

Furthermore, Park claims that he has calculated the function $\eta(\Delta/\xi)[F^2(b)/\kappa]$ more accurately[4] than we did[3] by taking the variation of the vector potential into account. This is not so, as he calculated this function with the assumption that $J=0$ which is the same as our approximation, for our values for $F^2(b)$ and Δ/ξ were taken from Ref. 11 (η is a parameter of order unity). Experiments[5,6] seem to agree better with our calculations[3] (with $\eta=1$) than with Park's.[4]

Equation (7) can be compared to the experimental results by Swartz and Hart.[12] They find for a polished well-annealed ribbon of $Pb_{0.95}Tl_{0.05}$ with cross-sectional dimensions $2a = 0.636$ cm, $a/b \approx 83$, a critical current of about 3.0 A at 4.2°K when $H_0 \approx H_{c2} = 1030$ G. The applied magnetic field H_0 was perpendicular to the current and parallel to the dimension $2a$. If we assume that H_c is that for Pb, namely, 545 G, it follows that $\kappa = H_{c2}/\sqrt{2}H_c = 1.34$, $(\Delta/\xi)F^2(b) \approx 0.90$, and $\lambda = [(\kappa/\sqrt{2}H_c)(\hbar c/2e)]^{1/2} = 7.58 \times 10^{-6}$ cm. With $\beta = 1$ the calculated critical current is 3.6 A which compares favorably with the measured value considering that values calculated from other theories are larger by a factor of 10 or more.

Hence, we may conclude that the critical sheath transport current in a long, rectangular foil is controlled by the larger dimension of the cross section when the applied magnetic field is parallel to this dimension. When this dimension approaches infinity the critical current density approaches zero.

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