tract No. AF 49(638)-1389.

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[1] C. G. Callan and S. B. Treiman, Phys. Rev. Letters 16, 153 (1966).

[2] Y. Hara and Y. Nambu, Phys. Rev. Letters 16, 875 (1966).

[3] See Eqs. (22) of Ref. 2.

[4] The nonvanishing of these amplitudes does not contradict the fact that $|\Delta I|=\frac{1}{2}$ forbids the physical decay $K^+\to\pi^++\pi^0$. This selection rule is based on the fact that the isotopic spins of π^+ and π^0 must by symmetrized according to the Bose statistics. In the amplitude calculated by PCAC, one of the pions is off the mass shell, while the other pion is still on the mass shell, so that there is no symmetry between two pions.

[5] N. Cabibbo and R. Gatto, Phys. Rev. Letters 8, 382 (1960).

[6] M. Suzuki, Phys. Rev. Letters 15, 986 (1965).

[7] By partial integration, Eq. (3) can be put into another form,

$$-f[(\Box\pi^{+*})\pi^0-\pi^{+*}(\Box\pi^0)]K^++\text{H.c.}, \qquad (3')$$

whose matrix element is identical with the first of Eq. (24) in Ref. 2. Although the small $\pi^+-\pi^0$ mass difference gives a nonvanishing contribution from (3′), it is better to keep the direct term $g\pi^{+*}\pi^0K^++\text{H.c.}$, in addition to (3).

[8] According to the second of Eqs. (24) in Ref. 2, the deviation of the extrapolated K_1^0 amplitude from the physical one is less than ~8%.

[9] In this context, ξ may be considered as representing the ratio of the octet to 27-plet amplitude.

[10] This was also pointed out by S. Oneda, Y. S. Kim, and D. Korff, Phys. Rev. 136, B1064 (1964). The conclusion remains the same, even if the $\pi^+-\pi^0$ mass difference is taken into account. The mass difference gives only the renormalization of the nonderivative coupling constant.

[11] This result remains unchanged, even if we consider the derivative coupling $\partial_\mu\pi^{+*}\partial_\mu K^+$.

[12] See the paper by Oneda, Kim, and Korff.[10]

[13] S. D. Drell, Springer Tracts in Modern Physics, Ergebnisse der exakten Naturwissenschaften, edited by G. Höhler (Springer-Verlag, Berlin, 1965), Vol. 39, p. 71.

[14] The $\eta\to2\gamma$ coupling is $1/\sqrt{3}$ times that of $\pi^0\to2\gamma$ from SU(3). Nothing is known about the $K^+\to\eta+\pi^+$ coupling. Since this process is allowed by $|\Delta I|=\frac{1}{2}$, one may assume this is as large as the $K_1^0\to2\pi$ coupling and the off-shell effect is negligible. From these considerations one concludes that the π^0 term will dominate the η term if $|\xi|\gg3$.

PION SCATTERING LENGTHS*

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The current commutation relations[1] and partially conserved axial-vector current (PCAC) assumption[2,3] allow the calculation of the matrix elements for emission and absorption of any number of soft pions[4] and, therefore, in particular, determine the scattering length of a pion on any target particle. In this note we give a simple formula for pion scattering on any particle but a pion,[5] and then extend this result to the more difficult case of pion-pion scattering.

Calculations of soft-pion matrix elements may be conveniently performed in three distinct steps:

Step I.—The S matrix is extended off the mass shell, using a pion field defined as proportional to the divergence of the axial-vector current. In our case we define the off-mass-shell invariant pion scattering amplitude $\langle f,qb|M|i,ka\rangle$ by

$$\int d^4x\,d^4y\langle f|T\{\partial_\mu A_b{}^\mu(x),\partial_\nu A_a{}^\nu(y)\}|i\rangle e^{-iqx}e^{iky}\equiv\frac{i(2\pi)^4\delta^4(p_f+q-p_i-k)F_\pi{}^2m_\pi{}^4}{(q^2+m_\pi{}^2)(k^2+m_\pi{}^2)(2\pi)^3(4E_iE_f)^{1/2}}\langle f,qb|M|i,ka\rangle, \qquad (1)$$

where k^μ and q^μ are the initial and final pion momenta, a and b are the initial and final pion isovector indices (running over 1, 2, 3), i and f label the initial and final states of the target particle, $A_a{}^\mu(x)$ is the axial-vector current, and F_π is the pion-decay amplitude, defined by

$$\langle 0|\partial_\nu A_a{}^\nu(0)|\pi_{qb}\rangle\equiv F_\pi m_\pi{}^2\delta_{ab}(2q^0)^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}}. \qquad (2)$$

Note that Eq. (1) is a definition, not a theorem or an assumption, but that the Lehmann-Symonzik-Zimmerman (LSZ) formalism[6] shows rigorously (and without invoking PCAC) that the S matrix is given in terms of M on the mass shell, by

$$\langle f,\vec{q}b|S|i,\vec{k}a\rangle = \frac{-i(2\pi)^4\delta^4(p_i+k-p_f-q)}{(2\pi)^6(16q^0k^0E_iE_f)^{1/2}} \times[\langle f,qb|M|i,ka\rangle]_{q^2=k^2=-m_\pi^2}. \quad (3)$$

Step II.—The current commutation relations are used to prove an exact theorem about the behavior of the matrix element in the limit of vanishing pion four-momenta. In our case it is convenient to fix $p_i^\mu=p^\mu$, and let q^μ and k^μ go to zero together, so that $p_f^\mu\to p^\mu$. [Since $p_f^2=-m_\pi^2$, we must require that $p\cdot k=p\cdot q$ to first order, but we do not necessarily take $q^\mu=k^\mu$.] The commutation rules used here are those suggested by the σ model[3,7] and the free-quark model:

$$\delta(x^0-y^0)[A_a^{\ 0}(y),A_b^{\ \mu}(x)] =2ig_v\epsilon_{abc}V_c^{\ \mu}(x)\delta^4(x-y)+\text{S.T.}, \quad (4)$$

$$\delta(x^0-y^0)[A_b^{\ 0}(x),\partial_\nu A_a^{\ \nu}(y)] =ig_V\sigma_{ab}(x)\delta^4(x-y)+\text{S.T.}, \quad (5)$$

where $V_c^{\ \mu}(x)$ is the vector current, $\sigma_{ab}(x)$ is some scalar field which may or may not have something to do with a real 0^+ π-π resonance or enhancement, and "S.T." means possible Schwinger terms. It will be assumed that the Schwinger terms are either c numbers, which do not contribute at all to the connected part of M, or, if operators, involve gradients which kill their contribution to first order in q and k. Our theorem states that, as q^μ and k^ν vanish, the connected part of M approaches

$$\langle f,qb|M|i,ka\rangle \to M_{fb,ia}^{(0)}-8(g_V/F_\pi)^2(p\cdot q)(T_\pi)_{ba}\cdot(T_t)_{fi} +\text{poles}+O(qq,qk,kk), \quad (6)$$

where $M^{(0)}$ is an unknown constant proportional to $\langle f|\sigma_{ab}(0)|i\rangle$, with $p_f=p_i=p$, and T_π and T_t are the pion and target isospin matrices, with $(T_{\pi c})_{ba}=i\epsilon_{abc}$. The "poles" in Eq. (6) are to be evaluated from the Born terms in gradient coupling theory[4]; for instance, there are no poles in π-π, π-K, or π-Λ scattering, while for π-N scattering the pole terms in Eq. (6) are

$$\text{"poles"}=\left(\frac{g_A}{F_\pi}\right)^2\left(\frac{m_N}{p\cdot q}\right)\bar{u}_f[\gamma_5\not{q}(-i\not{p}+m_N)\gamma_5\not{k}\tau_b\tau_a +\gamma_5\not{k}(-i\not{p}+m_N)\gamma_5\not{q}\tau_a\tau_b]u_i. \quad (7)$$

The proof follows standard lines. The left-hand side of Eq. (1) is identically equal to

$$\int d^4x\,d^4y\,e^{iky}\,e^{-iqx} \times\{-\delta(x^0-y^0)\langle f|[A_b^{\ 0}(x),\partial_\nu A_a^{\ \nu}(y)]|i\rangle -iq_\mu\delta(x^0-y^0)\langle f|[A_a^{\ 0}(y),A_b^{\ \mu}(x)]|i\rangle +q_\mu k_\nu\langle f|T\{A_b^{\ \mu}(x),A_a^{\ \nu}(y)\}|i\rangle\}. \quad (8)$$

Using Eqs. (4) and (5) and the known matrix elements of $V_c^{\ \mu}(x)$ at zero momentum transfer, the three terms of Eq. (8) yield, respectively, the first three therms of Eq. (6). Note that the first term of Eq. (8) does not produce an additional first-order term in Eq. (6), because Eq. (5) shows that it depends only upon $p\cdot(q-k)$ and $(q-k)^2$, and $p\cdot(q-k)=0$. The pole terms may be identified as the total first-order contribution of the last term in Eq. (8).

Step III.—The exact theorem proved in Step II is used to estimate the matrix element on the mass shell. It is here that we must for the first time invoke PCAC, by which we mean that the M defined in Step I is as smooth a function of q^μ and k^ν as would be expected in a perturbation expansion, based on a Lagrangian field theory in which $\partial_\mu A_a^{\ \mu}(x)$ is proportional to the pion field. [The statement, that $\partial_\mu A_a^{\ \mu}(x)$ is proportional to the pion field, is by itself empty.] In our present context we will interpret this rather Delphic hypothesis as meaning that, if the pole terms in Eq. (6) are understood to include the poles near $q=k=0$ (such as the 3-3 resonance in π-N scattering) as well as those actually at $q=k=0$ (such as the N pole itself), then the coefficients of the remaining quadratic terms in Eq. (6) are of order G_π^2/m_i^2, where m_i is some large internal mass, assumed to be of order m_N. Therefore, when the components of q^μ and k^ν in the rest frame of p^μ are of order m_π, the quadratic terms in Eq. (6) are of order $G_\pi^2 m_\pi^2/m_i^2$, while the Goldberger-Treiman relation shows that the linear terms are of order $G_\pi^2 m_\pi m_t/m_N^2$, where m_t is the mass of the target particle. Therefore, if the target mass is much larger than the pion mass, we may get a good approximation to the soft-pion S-matrix element by using Eq. (6) with quadratic and higher terms omitted. We can offer three arguments for also omitting the $M^{(0)}$ term:

(1) The Adler self-consistency argument[8] shows that M must vanish, except for poles, when $q^\mu = 0$ and $k^2 = -m_\pi^2$. Thus $M^{(0)}$ must be of the same order as the quadratic terms as this point, i.e., of order $G_\pi^2 m_\pi^2/m_i^2$, and is hence negligible. [This argument can be made more explicit by rearranging Eq. (8) to separate the one- and two-pion pole contributions to the last term, as was done in Ref. 4. We then find for M an expression like Eq. (6), but with $M^{(0)}$ multiplied by a factor $(q^2+k^2+m_\pi^2)/m_\pi^2$. The vanishing of this term at $q^\mu=0$, $k^2=-m_\pi^2$ is now automatic, and we see directly that if $M^{(0)}$ were large, it would contribute to M a rapidly varying function of q^μ and k^ν, in contradiction to the spirit of PCAC.][9]

(2) In the σ model,[3] $M^{(0)}$ is of order $G_\pi^2 m_\pi^2/m_\sigma^2$; this may be neglected if $m_\pi/m_t \ll (m_\sigma/m_N)^2$.

(3) The method of Ref. 4 can be used to show that, if $m_\pi = 0$ and $\partial_\mu A_a^{\ \mu} = 0$, then M obeys the limiting formula (6), but the $M^{(0)}$ absent. Hence $M^{(0)}$ may be regarded as arising only from the nonvanishing of the internal pion masses.

We still have the poles to consider, but these are generally absent in the s-wave part of the scattering amplitude. (This is true, for example, of the nucleon and 3-3 resonance poles in π-N scattering, the K^* poles in π-K scattering, etc.) Hence the $l=0$ part of the S matrix is given near threshold by just keeping the $p\cdot q$ term in Eq. (6), i.e.,

$$\langle f, \mathrm{q}b|S|i, \mathrm{k}a\rangle_{l=0} \cong [-i(g_V/F_\pi)^2/2\pi^2](T_\pi)_{ba}(T_t)_{fi} \times(-p\cdot q/m_\pi m_t)\delta^4(p_i+k-p_f-q).$$

The scattering length a_T is defined as $-2i\pi$ times the reduced mass times the coefficient of the δ function in S at threshold, so[5]

$$a_T = -L(1+m_\pi/m_t)^{-1}[T(T+1)-T_t(T_t+1)-2], \quad (9)$$

where T_t is the target isospin, T is the total isospin, and L is a convenient length, given by the Goldberger-Treiman relation as

$$L \equiv \frac{g_V^{\ 2} m_\pi}{2\pi F_\pi^{\ 2}} \simeq \frac{G_\pi^{\ 2} m_\pi}{8\pi m_N^{\ 2}}\left(\frac{g_V}{g_A}\right)^2 = 0.11 m_\pi^{\ -1}. \quad (10)$$

The reduced-mass correction $(1+m_\pi/m_t)^{-1}$ in Eq. (9) might well be omitted within the spirit of our approximations, but we keep it because it clearly arises from the definition of a_T.

For π-N scattering Eq. (9) gives

$$a_{1/2} = 2L(1+m_\pi/m_N)^{-1} = 0.20 m_\pi^{\ -1},$$

$$a_{3/2} = -L(1+m_\pi/m_N)^{-1} = -0.10 m_\pi^{\ -1}, \quad (11)$$

results which compare very well with the experimental values[10] $a_{1/2}m_\pi = 0.171 \pm 0.005$ and $a_{3/2}m_\pi = -0.088 \pm 0.004$. Using the prediction that $a_{1/2}-a_{3/2} = 3L(1+m_\pi/m_N)^{-1}$, together with the Goldberger-Miyazawa-Oehme sum rule[11] for $a_{1/2}-a_{3/2}$, we can obtain for (g_A/g_V) a sum rule, which differs from that of Adler and Weisberger[12] only in terms of order m_π^2/m_N^2; however, the sum rule is true only if the odd part of the forward scattering amplitude obeys an unsubtracted dispersion relation, while the derivation of the scattering lengths (11) made no assumption about the high-energy limit of any amplitude. The prediction that $a_{1/2}+2a_{3/2}$ is 0 may be regarded as a threshold version of Adler's self-consistency condition,[8] since we can easily see that if $M^{(0)}$ were nonnegligible, $a_{1/2}+2a_{3/2}$ would be proportional to $M^{(0)}$. In whatever form we choose to write the predictions (11), their success probably rules out the presence of any strong low-energy π-π interaction, for our derivation would have failed if M contained a strong singularity in the t channel at a mass near $2m_\pi$.

Equation (9) can also be used to calculate π-K and π-hyperon scattering lengths, but none of these have been measured yet. A few pion-nuclear scattering lengths have been measured[13] and do not compare well with Eq. (9), but this is presumably because 140 MeV is such a high excitation energy for nuclei that we cannot regard a pion at threshold as soft; in particular, Eq. (9) is real, while in fact pion annihilation makes a_T complex. This point is under further study.

We derived Eq. (9) under the assumption that the target is much heavier than a pion, so Eq. (9) cannot be used for pion-pion scattering. For instance, Eq. (9) gives a nonvanishing scattering length for $T=1$, in contradiction with Bose statistics. In order to calculate the π-π scattering lengths we will have to keep track of $M^{(0)}$ and the qq, qk, and kk terms, because at threshold they are here just as large as the pq term.

First note that crossing symmetry, isospin conservation, and Bose statistics require that the expansion of the off-mass-shell π-π scattering amplitude to second order in momenta takes the form[14]

$$\begin{aligned}\langle ld, qb|M|pc, ka\rangle &= \delta_{ab}\delta_{cd}[A+B(s+u)+Ct+\cdots]\\ &+\delta_{ad}\delta_{cb}[A+B(s+t)+Cu+\cdots]\\ &+\delta_{ac}\delta_{bd}[A+B(u+t)+Cs+\cdots],\end{aligned} \tag{12}$$

where

$$s\equiv-(p+k)^2,\quad t\equiv-(k-q)^2,\quad u\equiv-(p-q)^2.$$

Also A, B, and C are constant coefficients, and "$+\cdots$" denotes terms of fourth and higher order in the pion four-momenta p, k, l, and q. The crucial point about Eq. (12) is that there is no way that M can contain terms linear in the masses $-p^2, -l^2, -q^2, -k^2$, aside from the terms proportional to s, t, and u.

The PCAC assumption says that if M is defined in analogy with Eq. (1) as proportional to the Fourier transform of the vacuum expectation value of the time-ordered product of four axial-vector divergences, then the quartic and higher order terms in M are small when p^μ, l^μ, k^μ, and q^μ are of order m_π or less. The physical threshold is at $s=4m_\pi^2$, $t=u=0$, so that π-π scattering lengths are

$$a_0\cong-(1/32\pi m_\pi)[5A+8m_\pi^2B+12m_\pi^2C], \tag{13}$$

$$a_2\cong-(1/32\pi m_\pi)[2A+8m_\pi^2B]. \tag{14}$$

Equation (6) shows that when $p^\mu=l^\mu$ is on the mass shell and $q^\mu=k^\mu\to0$, the matrix element approaches

$$\begin{aligned}\langle ld, qb|M|pc, ka\rangle &\to M_{db,ca}^{(0)}-8(g_V/F_\pi)^2(p\cdot q)\\ &\times(\delta_{da}\delta_{bc}-\delta_{bd}\delta_{ac}).\end{aligned} \tag{15}$$

(There are no poles here.) In this limit $t=0$, $s\to m_\pi^2-2p\cdot q$, and $u\to m_\pi^2+2p\cdot q$, so comparing Eq. (15) with Eq. (12) gives

$$B-C=4(g_V/F_\pi)^2, \tag{16}$$

$$\begin{aligned}M_{db,ca}^{(0)} &= \delta_{ab}\delta_{cd}[A+2m_\pi^2B]\\ &+(\delta_{ad}\delta_{bc}+\delta_{bd}\delta_{ac})[A+m_\pi^2C+m_\pi^2B].\end{aligned} \tag{17}$$

From Eqs. (13), (14), (16), and (10), we find[15]

$$2a_0-5a_2=6L=0.69m_\pi^{-1}. \tag{18}$$

The Adler self-consistency argument[8] shows that M vanishes when any one of the four pion momenta vanish and the other three are on the mass shell, i.e., $M=0$ when $s=t=u=m_\pi^2$, so

$$A=-m_\pi^2(2B+C). \tag{19}$$

In order to use this result to get another relation for the scattering lengths, it is necessary to add a little new physical information. If we assume that $\partial_\mu A_a{}^\mu$ is part of a chiral quadruplet along with an isoscalar field (as in the σ model or the free-quark model), then Eq. (5), and hence $M_{db,ca}^{(0)}$ is proportional to δ_{ba}, so Eq. (17) gives

$$A=-m_\pi^2(B+C). \tag{20}$$

From Eqs. (19) and (20) we have then

$$B=0,\quad A=-m_\pi^2C \tag{21}$$

which, with Eqs. (13) and (14), yields

$$a_0/a_2=-\tfrac{7}{2}. \tag{22}$$

Combining Eqs. (18) and (22), we find

$$a_0=(7/4)L=0.20m_\pi^{-1},\quad a_2=-\tfrac{1}{2}L=-0.06m_\pi^{-1}, \tag{23}$$

and the full low-energy π-π matrix element is

$$\langle ld, qb | M | pc, ka \rangle = 4(g_V/F_\pi)^2\{[m_\pi^2-t]\delta_{ab}\delta_{cd} + [m_\pi^2-u]\delta_{ad}\delta_{bc} + [m_\pi^2-s]\delta_{ac}\delta_{bd}\}. \quad (24)$$

The striking feature of our result (23) is, of course, that a_0 is predicted to be very much smaller than anyone had thought. It seems appropriate, therefore, to close with some remarks about the theoretical and experimental plausibility of this result:

(1) In the σ model[3] the $\lambda\varphi_\pi^4$ term plus the three one-σ-exchange graphs gives scattering lengths which agree precisely with Eq. (23) in the limit $m_\sigma^2 \gg m_\pi^2$, except that L is given in terms of unrenormalized coupling constants.

(2) If the odd part of the forward π-π scattering amplitude obeys an unsubtracted dispersion relation, then our prediction (18) of $2a_0-5a_2$ may be used to derive the Adler sum rule[16] for π-π scattering total cross sections. (We have already made the corresponding remark for π-N scattering, that our prediction for $a_{1/2}-a_{3/2}$ is essentially equivalent to the Adler-Weisberger relation.) This sum rule seems to require either that a_0 be large, or that there exist a strong π-π resonance. The σ model provides one example where it is a resonance (of arbitrary mass) rather than a large scattering length, that saturates the sum rule.

(3) We have already remarked that the success of our prediction (11) of the π-N scattering lengths (in particular, the prediction $a_{1/2} + 2a_{3/2} = 0$) would be difficult to understand, if there were any strong low energy π-π interaction. In the same way, the success of a recent calculation[17] of the two K_{e4} form factors provides further experimental evidence that the π-π scattering lengths are quite small.

(4) Experiments on τ and η decay and high-energy "peripheral" two-pion production are ambiguous, since it is not clear whether the effects seen have anything to do with a π-π interaction. Furthermore, the two-pion production, even if peripheral, measures the π-π amplitude with one pion off the mass shell, and the conditions (19) or (20) show that this has a very large effect on M. It would seem desirable to reanalyze these experiments, using formulas (12) or (24) for the off-mass-shell π-π amplitude.

(5) A strong low-energy π-π interaction would have a large effect on the process $\pi+N \to 2\pi+N$ at threshold, where all three pions are soft. A calculation of this process is now under way, in conjunction with Chang.

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[1]M. Gell-Mann, Physics 1, 63 (1964).

[2]Y. Nambu, Phys. Rev. Letters 4, 380 (1960).

[3]M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960).

[4]S. Weinberg, Phys. Rev. Letters 16, 879 (1966).

[5]While preparing this note I became aware that Y. Tomozawa (to be published) had earlier calculated π-N, π-K, and π-hyperon scattering lengths, with results in agreement with our general formula (9). Since then similar results have also been obtained by B. Hamprecht (to be published), K. Raman and E. C. G. Sudarshan (to be published), and by A. P. Balachandran, M. Gundzik, and F. Nicodemi (to be published). I have nevertheless decided to present the derivation of Eq. (9) here, with a fuller attention to the details of the extrapolation onto the mass shell, partly to make it clear that Eq. (9) applies to the scattering of pions on all elementary particles except pions, and partly to serve as a basis for the π-π calculation, which the above authors do not attempt.

[6]H. Lehmann, K. Symanzik, and W. Zimmerman, Nuovo Cimento 1, 205 (1955).

[7]J. Schwinger, Ann. Phys. (N.Y.) 2, 407 (1957). By using a Jacobi identity and the conservation of $V_c^\mu(x)$, we can show that $\sigma_{ab}(x)$ is always symmetric in a and b, while in the σ model or free-quark model, it is proportional to σ_{ab}. It is only in the case of π-π scattering that we need distinguish different forms of this commutator.

[8]S. L. Adler, Phys. Rev. 137, B1022 (1965); 139, B1638 (1965). Adler uses this argument to give a formula for one of the two functions (A) entering in the symmetric π-N forward scattering amplitude, and then has to use a dispersion relation to compare this result with experiment. It can be seen directly from Adler's formula for A that the full symmetric amplitude vanishes at threshold, A being just cancelled by the pole in the other function B. (Indeed, this could have been seen immediately from the facts that M obviously vanishes as $q \to 0$ gradient coupling theory,[3] and gradient coupling theory satisfies PCAC.) It is clearer to deal with the full amplitude, rather than to divide it into A and B terms, and it is certainly simpler to work at threshold where nucleon and 3-3 resonance poles make no contribution, than to have to use dispersion theory to account for their contribution below threshold. For example, G. F. Chew has reminded me that the relation $f_{1/2} = -2f_{3/2}$, which works beautifully for the forward scattering amplitudes at

threshold, does not work at all well below threshold, presumably because the 3-3 resonance makes a large p-wave contribution except at threshold.

[9]Similar conclusions have been reached by M. L. Goldberger and S. B. Treiman, private communication.

[10]J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. 35, 737 (1963).

[11]M. L. Goldberger, H. Miyazawa, and R. Oehme, Phys. Rev. 99, 986 (1955).

[12]S. L. Adler, Phys. Rev. Letters 14, 1051 (1965); W. I. Weisberger, Phys. Rev. Letters 14, 1047 (1965). In some derivations of the Adler-Weisberger sum rule, a formula is first obtained for the antisymmetric part of the forward pion-nucleon scattering amplitude at $s=m_N^2$ and zero external pion mass; see, e.g., Eq. (49) of Ref. 16. It was originally thought that this formula could only be compared with experiment by using it to derive a sum rule; our point here is that it can be converted into a formula for $a_{1/2}-a_{3/2}$ by direct use of PCAC.

[13]R. Seki and A. H. Cromer, to be published.

[14]The expansion Eq. (12) is certainly no good in the physical region well above threshold, since unitarity requires the presence in M of odd powers of $i(s-4m_\pi^2)^{1/2}$. However, the scattering lengths we predict are quite small, so it is at least self-consistent to suppose that the unitarity branch point is a weak singularity, which does not prevent our using Eq. (12) up to and somewhat beyond threshold. An analogous expansion is known to work well in the similar case of τ decay. [For rigorous results concerning analyticity in s, t, and u when all external masses are equal, see A. Minguzzi, J. Math. Phys. 7, 679 (1966).] In particular, even where Eq. (12) begins to conflict seriously with unitarity, it would seem quite reasonable to apply it to the real part of the invariant matrix element. These points may in principle be checked by measuring the p-wave scattering length a_1 and the s-wave effective ranges r_0 and r_2, defined by the relation $k^{2l+1}\cot\delta_T \to a_T+\frac{1}{2}k^2 r_T$. In the physical region, Eq. (12) gives the matrix element in terms of just two parameters $A+4m_\pi^2B$ and $B-C$; hence, without using any other assumptions, we find that if Eq. (12) holds up to threshold, the scattering lengths must be subject to one relation, which may be written $18m_\pi^2a_1=2a_0-5a_2$. If we also believe that Eq. (12) holds for the real part of M somewhat beyond threshold, then we also find that $6m_\pi^2a_0^2(a_0+\frac{1}{2}r_0)=a_0+10a_2$ and $6m_\pi^2a_2^2\times(a_2+\frac{1}{2}r_2)=5a_0-5a_2$. Similar remarks were made by G. F. Chew and S. Mandelstam, Nuovo Cimento 19, 752 (1960). Note that keeping just the zeroth-order term A in Eq. (12) would yield $a_0/a_2=\frac{5}{2}$ and $a_1=0$, but this would be a very bad approximation because both Eqs. (19) and (20) show that A is of the same order as m_π^2B or m_π^2C.

[15]A similar result is derived by F. T. Meire and M. Sugawara (to be published) but without justification of the neglect of mass-extrapolation terms. They then set $a_2=0$, and hence get a value of a_0 twice as large as ours.

[16]S. L. Adler, Phys. Rev. 140, B736 (1965).

[17]S. Weinberg, Phys. Rev. Letters 17, 336 (1966).