all be the same. It seems likely that crossover of quarks from one hadron to another during a collision will be facilitated by making the pair-interaction ranges slightly greater than the three-particle ranges and adjusting the strength accordingly. (d) The model can be used for explicit calculations of hadron processes by assuming particular forms for the space dependences of the two- and three-particle interactions, so that they lead to the potential energy given by (1) and (4). (e) There must be an additional potential energy that is much weaker than (1) and of longer range, which is responsible for the bulk of the interactions between hadrons.

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EXTRAPOLATION OF THE AMPLITUDES OF NONLEPTONIC DECAYS OF K MESONS*

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In the calculation of nonleptonic decay amplitudes of K mesons from partially conserved axial-vector current (PCAC) and the current algebra of SU(3)⊗SU(3), it has been recognized that large effects are to be expected in extrapolating from the physical amplitudes with the pions on their mass shells to the off-shell amplitudes evaluated at the vanishing pion momenta[1,2] (hereafter called "off-shell effect"). In this note we derive a simple extrapolation formula under certain assumptions, and apply it to the decay $K^+\to\pi^++2\gamma$ with two photons having the total energy different from the π^0 mass.

The most striking example of large off-shell effects is found in the $K\to 2\pi$ decays[3]:

$$M[K^+\to\pi^++\pi^0;\, q(\pi^+)=0]=-M[K^+\to\pi^++\pi^0;\, q(\pi^0)=0], \tag{1}$$

$$=iM[K_1^0\to\pi^++\pi^-;\, q(\pi^+)=0]=iM[K_1^0\to\pi^++\pi^-;\, q(\pi^-)=0], \tag{2}$$

where $q(\pi^+)$, for example, denotes the four-momentum of π^+, and M, the decay amplitude. Namely, the amplitude of the K^+ decay calculated by applying PCAC to π^+ has the opposite sign from that calculated by PCAC applied to π^0. Neither amplitude necessarily vanishes even under the $|\Delta I|=\frac{1}{2}$ rule,[4] and they are equal in magnitude to that of the K_1^0 decay. Experimentally, the $K_1^0\to\pi^++\pi^-$ decay amplitude is ~20 times larger than the $K^+\to\pi^++\pi^0$ decay amplitude. Therefore, the off-shell effect must be substantial in either or both of these decays.

However, this does not necessarily mean that the form of the analytic continuation of these amplitudes is hopelessly complicated. In fact, Hara and Nambu[2] suggested an extrapolation formula which is linear in the squared momenta of the pions. One of the purposes of this note is to show that such an approximation can be justified by rather simple assumptions. It is also interesting to find that this linear approximation is equivalent to introducing the derivative coupling,

$$f[\pi^{+*}(\overleftarrow{\partial}_\mu-\overrightarrow{\partial}_\mu)\pi^0]\partial_\mu K^+ +\text{H.c.}, \tag{3}$$

which, first considered by Cabibbo and Gatto in relation to the radiative K^+ decay,[5] satisfies $|\Delta\,|=\frac{1}{2}$ but vanishes on mass shells. The antisymmetric property shown in (1) is clearly seen in (3). This argument will be a good answer to a possible criticism that it is difficult to imagine a dynamical mechanism for a large off-shell effect. The origin of this off-shell effect is rather kinematical, based on the Bose statistics of the pions.

The amplitude of the decay $K^+ \to \pi^+ + \pi^0$, for example, is given by

$$(4\omega_+\omega_K)^{-\frac{1}{2}} M(q^2) = (q^2 + \mu^2) \int d^4x\, e^{-iqx} \times T\langle \pi^+ | \varphi^{(0)}(x) H_W(0) | K^+ \rangle, \quad (4)$$

where H_W is the weak Hamiltonian, and q, μ, and $\varphi^{(0)}$ are the momentum, mass, and field operator of π^0, respectively. We omitted explicit dependence of M on other variables [$q^2(\pi^+)$ and $q^2(K^+)$], because we are, at the moment, interested in how M depends on $q^2 = q^2(\pi^0)$. By using PCAC and integrating by parts, one gets

$$M(q^2) = c^{-1}(q^2 + \mu^2)[R_1(q^2) + R_2(q^2)], \quad (5)$$

where

$$R_1(q^2) = \int d^3x\, e^{-i\vec{q}\cdot\vec{x}} \times \langle \pi^+ | [A_0^{(3)}(\vec{x}, 0), H_W(0)] | K^+ \rangle,$$

$$R_2(q^2) = iq_\mu \int d^4x\, e^{-iqx} \times T\langle \pi^+ | A_\mu^{(3)}(x) H_W(0) | K^+ \rangle. \quad (6)$$

Here $c = (\sqrt{2} m_N \mu^2 / g_{NN\pi})(-G_A/G_V)$, and $A_\mu^{(i)}$ is the axial-vector current with the isospin index, i.

One can prove a theorem about the equal-time commutator term $R_1(q^2)$. Theorem: $R_1(q^2)$ has no pion pole $(q^2 + \mu^2)^{-1}$. Proof: First note that $\vec{q}$ appearing in the exponential factor is trivial, because the equal-time commutator is assumed to be proportional to $\delta(\vec{x})$. Then, according to Suzuki,[6] the right-hand side of $R_1(q^2)$ in Eq. (6) is the matrix element of a parity-conserving density. Therefore, $R_1(q^2)$ is formally equivalent to the amplitude of the process $K^+ \to \pi^+ + \sigma$, where σ is a (fictitious) neutral scalar meson, which carries a momentum q and whose interaction is parity conserving. On the other hand, if $R_1(q^2)$ has a pion pole $(q^2 + \mu^2)^{-1}$, it must be associated with the "pole diagram" corresponding to the process $K^+ \to \pi^+ + \pi^0$ followed by $\pi^0 \to \sigma$, which violates parity. (QED.)

The above proof suggests also that $R_1(q^2)$ can be considered a slowly varying function of q^2 unless a scalar meson exists with the mass close to the pion mass.

The so-called surface term $R_2(q^2)$ has a pion pole, which can be estimated to be

$$R_2(q^2)_{\text{pole}} = -\frac{c}{\mu^2} \frac{q^2}{q^2 + \mu^2} M(-\mu^2). \quad (7)$$

If we neglect the contribution from the higher mass states, substitution of (7) into (5) gives

$$M(q^2) = c^{-1}(q^2 + \mu^2) R_1(q^2) - (q^2/\mu^2) M(-\mu^2). \quad (8)$$

Putting $q^2 = 0$, we have

$$M(0) = c^{-1}\mu^2 R_1(0).$$

This equation differs from the usual and exact formula by which the equal-time commutator term gives the amplitude extrapolated to $q_\mu(\pi^0) = 0$, or equivalently, to $q^2(\pi^0) = 0$, together with $q^2(\pi^+) = q^2(K^+)$. The difference comes from the higher mass contribution, which may have the term proportional to $q[q(K^+) - q(\pi^+)] = q^2(K^+) - q^2(\pi^+)$. Thus under the assumption that the higher mass contribution is completely negligible and that no scalar meson exists near the considered range of q^2, one can derive from (8) a linear formula,

$$M(q^2) = M(-\mu^2)[1 - \xi(q^2 + \mu^2)/\mu^2], \quad (9)$$

with

$$\xi = [M(-\mu^2) - M(0)]/M(-\mu^2).$$

It should be emphasized that the validity of this approximation is not destroyed simply because $|\xi|$ is much larger than unity.

It is easy to see that the matrix element of (3) with π^+ and K^+ on their mass shells is identical with the second term of (9) if[7]

$$f = \xi\mu^{-2} M(-\mu^2) = \mu^{-2}[M(-\mu^2) - M(0)].$$

The same procedure can also be applied to the K_1^0 decay, and it is possible that this decay amplitude has a large off-shell effect. However, because of the relations (1) and (2), the maximum off-shell effect is expected for the K^+ decay by assuming that the off-shell effect is small in the K_1^0 decay.[8] Then the extrapolated amplitudes in (2) are given, to a good approximation, by the physical K_1^0 decay amplitude, and one gets

$$\left| \frac{M(0)}{M(-\mu^2)} \right| \approx \left[\frac{\Gamma(K_1^0 \to \pi^+ + \pi^-)}{\Gamma(K^+ \to \pi^+ + \pi^0)} \right]^{1/2} \approx 20,$$

therefore,

$$|\xi| \approx 20,$$

for the decay $K^+ \to \pi^+ + \pi^0$.[9] Of course, additional off-shell effects are to be expected from conventional dynamics. These are, however, usually supposed to be small, especially in the spirit of renormalization theory. If ξ is large enough, the contribution considered here will certainly dominate the total off-shell effects.

It is interesting to look for physical processes in which this extremely large but well-defined off-shell effect manifests itself. For this purpose we first consider the process $K^+ \to \pi^+ + \pi^0 + \gamma$. By using the form (9), we find that the off-shell term certainly affects the pion current contribution. However, gauge invariance requires us to consider the "contact term" obtained by replacing q_μ in (9) by $q_\mu + e\mathfrak{A}_\mu$, and a straightforward calculation shows that this contribution cancels completely the ξ-dependent part of the pion current term. The calculation is exactly the same as in Ref. 5 using (3). Unfortunately its Eq. (4) contains an error: The f term must vanish.[10]

Next we consider the process $K^+ \to \pi^+ + 2\gamma$. One can expect that the off-shell term of the $K^+ \to \pi^+ + \pi^0$ vertex (9) appearing in the process

$$K^+ \to \pi^+ + (\pi^0)_{\text{virtual}} \quad \lfloor\!\to 2\gamma, \tag{10}$$

will contribute to the part of the decay spectrum in which the total energy of the two photons is different from the π^0 mass. Among alternative mechanisms leading to the same final state, we first consider the internal bremsstrahlung effect. This estimation can be done by introducing the effective weak interaction of the form $h\pi^{+*}K^+$. A direct calculation with proper consideration of gauge invariance shows that the total contribution from the π^+ and K^+ currents vanishes.[11] Another similar contribution will come from the process, $K^+ \to \pi^+ \to \rho^+ + \gamma$ followed by $\rho^+ \to \pi^+ + \gamma$, or $K^+ \to K^{*+} + \gamma \to K^+ + \gamma + \gamma$ followed by $K^+ \to \pi^+$. By determining the weak coupling constant h from SU(3) and the K_1^0-K_1^2 mass difference[12] and using a "moderate" estimate $\Gamma(\rho^+ \to \pi^+ + \gamma) \approx \Gamma(K^{*+} \to K^+ + \gamma) \approx 0.15$ MeV,[13] one finds that the branching ratio of these contributions to the total K^+ decay rate is negligibly small ($\sim 0.3 \times 10^{-7}$; compare this with the ratios in Table I).

Now the decay spectrum of the process (10) is calculated as a function of the total energy λ of the two photons:

$$\frac{d\Gamma_1(\lambda)}{d\lambda} = \frac{2}{\pi}\Gamma(\pi^0 \to 2\gamma)\Gamma(K^+ \to \pi^+ + \pi^0)\frac{Q(\lambda)\lambda^5}{Q(\mu)\mu^7} f(\lambda^2), \tag{11}$$

Table I. The branching ratio of the decay $K^+ \to \pi^+ + 2\gamma$, where the total energy of the two photons is larger than $1.5\,\mu$. Upper and lower values correspond to positive and negative ξ, respectively.

$\lvert\xi\rvert$	0	1	4	20
$\frac{\overline{\Gamma}_1}{\Gamma(K^+ \to \text{all})} \times 10^7$	0.081	1.18 0.32	12.5 9.01	275 258

where

$$Q(\lambda) = (1/2m_K)\{[m_K^{\,2} - (\lambda+\mu)^2][m_K^{\,2} - (\lambda-\mu)^2]\}^{1/2},$$

and

$$f(\lambda^2) = \left| \frac{1}{\lambda^2 - \mu^2 + i\epsilon} + \frac{\xi}{\mu^2} \right|^2,$$

with $\epsilon = \mu\Gamma(\pi^0 \to 2\gamma)$. The spectrum has a large peak centered at $\lambda = \mu$, and a smooth part off the pion mass. For our purpose it is necessary to observe the photons with λ far from μ. The factor λ^5 in (11) gives another peak (at $\lambda \sim 2.4\,\mu$) between the pion mass and the high-energy end $\lambda = m_K - \mu$. Therefore, as a tentative estimate, we calculate the partial width $\overline{\Gamma}_1$ integrated over the interval

$$1.5\,\mu \leq \lambda \leq m_K - \mu \cong 2.54\,\mu,$$

which corresponds to the range $0 \sim 83$ MeV for the kinetic energy of π^+. In Table I the branching ratio $\overline{\Gamma}_1/\Gamma(K^+ \to \text{all})$ is given for various values of ξ. These ratios might be within the reach of the present or near future experiment if $\lvert\xi\rvert \gtrsim 4$.

In (11) we have omitted several other possible terms giving similar results. For example, the other vertex, $\pi^0 \to 2\gamma$, may have a similar q^2 dependence, and the η meson must also be taken into account.[14] However, one can hopefully expect that the contribution from the off-shell term of the $K^+ \to \pi^+ + \pi^0$ vertex would dominate the whole process as long as $\lvert\xi\rvert$ is sufficiently larger than unity. Also the higher mass contribution to $R_2(q^2)$ may not be negligible for the range of q^2 considered. It is, however, quite unlikely that such contributions would cancel most of the effect already considered.

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[2]Y. Hara and Y. Nambu, Phys. Rev. Letters 16, 875 (1966).

[3]See Eqs. (22) of Ref. 2.

[4]The nonvanishing of these amplitudes does not contradict the fact that $|\Delta I|=\frac{1}{2}$ forbids the physical decay $K^+\rightarrow\pi^++\pi^0$. This selection rule is based on the fact that the isotopic spins of π^+ and π^0 must by symmetrized according to the Bose statistics. In the amplitude calculated by PCAC, one of the pions is off the mass shell, while the other pion is still on the mass shell, so that there is no symmetry between two pions.

[5]N. Cabibbo and R. Gatto, Phys. Rev. Letters 8, 382 (1960).

[6]M. Suzuki, Phys. Rev. Letters 15, 986 (1965).

[7]By partial integration, Eq. (3) can be put into another form,

$$-f[(\Box\pi^{+*})\pi^0-\pi^{+*}(\Box\pi^0)]K^++\text{H.c.}, \tag{3'}$$

whose matrix element is identical with the first of Eq. (24) in Ref. 2. Although the small $\pi^+-\pi^0$ mass difference gives a nonvanishing contribution from (3′), it is better to keep the direct term $g\pi^{+*}\pi^0K^++\text{H.c.}$, in addition to (3).

[8]According to the second of Eqs. (24) in Ref. 2, the deviation of the extrapolated K_1^0 amplitude from the physical one is less than ~8%.

[9]In this context, ξ may be considered as representing the ratio of the octet to 27-plet amplitude.

[10]This was also pointed out by S. Oneda, Y. S. Kim, and D. Korff, Phys. Rev. 136, B1064 (1964). The conclusion remains the same, even if the $\pi^+-\pi^0$ mass difference is taken into account. The mass difference gives only the renormalization of the nonderivative coupling constant.

[11]This result remains unchanged, even if we consider the derivative coupling $\partial_\mu\pi^{+*}\partial_\mu K^+$.

[12]See the paper by Oneda, Kim, and Korff.[10]

[13]S. D. Drell, Springer Tracts in Modern Physics, Ergebnisse der exakten Naturwissenschaften, edited by G. Höhler (Springer-Verlag, Berlin, 1965), Vol. 39, p. 71.

[14]The $\eta\rightarrow 2\gamma$ coupling is $1/\sqrt{3}$ times that of $\pi^0\rightarrow 2\gamma$ from SU(3). Nothing is known about the $K^+\rightarrow\eta+\pi^+$ coupling. Since this process is allowed by $|\Delta I|=\frac{1}{2}$, one may assume this is as large as the $K_1^0\rightarrow 2\pi$ coupling and the off-shell effect is negligible. From these considerations one concludes that the π^0 term will dominate the η term if $|\xi|\gg 3$.

PION SCATTERING LENGTHS*

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The current commutation relations[1] and partially conserved axial-vector current (PCAC) assumption[2,3] allow the calculation of the matrix elements for emission and absorption of any number of soft pions[4] and, therefore, in particular, determine the scattering length of a pion on any target particle. In this note we give a simple formula for pion scattering on any particle but a pion,[5] and then extend this result to the more difficult case of pion-pion scattering.

Calculations of soft-pion matrix elements may be conveniently performed in three distinct steps:

Step I.—The S matrix is extended off the mass shell, using a pion field defined as proportional to the divergence of the axial-vector current. In our case we define the off-mass-shell invariant pion scattering amplitude $\langle f,qb|M|i,ka\rangle$ by

$$\int d^4x\,d^4y\langle f|T\{\partial_\mu A_b{}^\mu(x),\partial_\nu A_a{}^\nu(y)\}|i\rangle e^{-iqx}e^{iky}\equiv\frac{i(2\pi)^4\delta^4(p_f+q-p_i-k)F_\pi{}^2m_\pi{}^4}{(q^2+m_\pi{}^2)(k^2+m_\pi{}^2)(2\pi)^3(4E_iE_f)^{1/2}}\langle f,qb|M|i,ka\rangle, \tag{1}$$

where k^μ and q^μ are the initial and final pion momenta, a and b are the initial and final pion isovector indices (running over 1, 2, 3), i and f label the initial and final states of the target particle, $A_a{}^\mu(x)$ is the axial-vector current, and F_π is the pion-decay amplitude, defined by

$$\langle 0|\partial_\nu A_a{}^\nu(0)|\pi_{qb}\rangle\equiv F_\pi m_\pi{}^2\delta_{ab}(2q^0)^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}}. \tag{2}$$