REGGE POLES IN UNEQUAL-MASS SCATTERING PROCESSES*

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Recent experimental and theoretical work indicates that the Regge-pole theory is important in the description of high-energy πN backward scattering.[1-4] However, the question of whether the Regge asymptotic form $s^{\alpha(u)}$ holds in the backward region has never been settled because there is a cone about the backward direction in which $\cos\theta_u$ does not become large with increasing s. And there seems to be general uneasiness[5,6] in applying the Regge asymptotic form in this region.

We have studied and resolved this kinematic ambiguity of the Regge representation, and in this note outline our argument and discuss some very interesting features of the unequal-mass scattering problem and of the Regge-pole theory in general, which our investigations have revealed. In brief, we find that the Regge form $s^{\alpha(u)}$ does hold throughout the backward region, but in order to cancel singularities which would otherwise appear at $u=0$, Regge trajectories must exist in families whose $u=0$ intercepts differ by integers. We discuss some experimental implication of this idea. Further, we are able to characterize the behavior of partial-wave amplitudes $a(u,l)$ at $u=0$ and find results in contradiction with those commonly believed.[7] A more detailed paper on this subject will be published elsewhere.[8]

Usual discussions[5] of the asymptotic behavior in the backward region are based on the application of the Sommerfeld-Watson transformation to expansions of the scattering amplitude in partial waves in the u channel. The high-energy limit is introduced through the variable

$$z_u = \cos\theta_u = -\left[1+\frac{2[su-(m^2-\mu^2)^2]}{u^2-2u(m^2+\mu^2)+(m^2-\mu^2)^2}\right]. \quad (1)$$

This variable is bounded by unity for all s in the backward cone defined by $0 \leq u \leq u_B = (m^2-\mu^2)^2 s^{-1}$, and, since z_u does not become large with increasing s, the conventional Regge representation (i.e., the Sommerfeld-Watson transformed partial-wave expansion) does not furnish an asymptotic limit in this region. Indeed any representation $A(u,s)=g(u,z_u)$ of the scattering amplitude is suspicious at $u=0$ because the transformation of variables is singular there.

Our method is based on work of Khuri[9] who shows that Sommerfeld-Watson transformations and Regge analysis can be applied to representations other than partial-wave expansions. Starting from a power series in the momentum transfer t, we establish a representation which explicitly exhibits Regge behavior throughout the backward region.

In our notation u is always the Regge-pole channel. For mathematical simplicity we treat the case of two spinless particles with masses m and μ, $\mu < m$, and assume that the third-channel spectral function $A_s(u,s)\equiv 0$. The more realistic case $A_s \neq 0$ is fully treated in Ref. 8. The method can also be generalized to include spin and definitely applies to πN scattering.

We assume that the ordinary partial-wave amplitudes $a(u,l)$ are meromorphic in the half-plane $\mathrm{Re}\,l > -\frac{1}{2}$, so that a Regge representation can be written for the scattering amplitude

$$A(u,t)=\frac{i}{2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl\,(2l+1)a(u,l)\frac{P_l(-z_u)}{\sin\pi l} -\pi\sum_i \beta_i(u)[2\alpha_i(u)+1]\frac{P_{\alpha_i(u)}(-z_u)}{\sin\pi\alpha_i(u)}, \quad (2)$$

where the sum is over the finite number of Regge trajectories to the right of background.

The amplitude $A(u,t)$ is in a disk of radius $t_0=4\mu^2$ about the origin in the t plane. We can express it as a power series

$$A(u,t)=\sum_{\nu=0}^{\infty} b(u,\nu)t^{\nu} \quad (3)$$

with coefficients

$$b(u,\nu)=\pi^{-1}\int_{t_0}^{\infty}dt\,A_t(u,t)t^{-\nu-1}, \quad (4)$$

where $A_t(u,t)$ is the spectral function in the momentum-transfer dispersion relation. Actually, the integral defining $b(u,\nu)$ converges only for $\mathrm{Re}\,\nu > N$, where N is the number of subtractions necessary in the dispersion relation, and must be defined by analytic continuation to the left of this line. For $\mathrm{Re}\,\nu > N$, $b(u,\nu)$

is analytic in ν and has only the physical cut in u.

The Regge representation is valid for $u \neq 0$, and we use it to compute A_t and in this way determine the continuation of $b(u,\nu)$ to the left of $\mathrm{Re}\,\nu = N$:

$$A_t(u,t) = D(u,t) + \sum_i \beta_i(u)[2\alpha_i(u)+1]P_{\alpha_i(u)}(1+t/2q^2). \quad (5)$$

$D(u,t)$ is the discontinuity of the Regge background integral and is of order $0(t^{-1/2})$ for $u \neq 0$. Its contribution to $b(u,\nu)$ through (4) is, therefore, analytic in $\mathrm{Re}\,\nu > -\frac{1}{2}$.

The contribution of the Regge-pole terms can be found from the integrals

$$\int_{t_0}^{\infty} dt\, t^{-\nu-1} P_{\alpha(u)}(1+t/2q^2). \quad (6)$$

Khuri[9] has shown that (6) is regular for $\mathrm{Re}\,\nu > -\frac{1}{2}$ except for simple poles at $\nu = \alpha(u)$, $\alpha(u)-1, \cdots, \alpha(u)-n$, where $\frac{1}{2} > \mathrm{Re}\,\alpha(u)-n > -\frac{1}{2}$. Thus the image of a single Regge pole is a principal Khuri pole at $\nu = \alpha(u)$ plus satellite poles displaced to the left by integers. The residues of the Khuri poles have been computed in Refs. 8 and 9. We can write for $b(u,\nu)$ the representation

$$b(u,\nu) = \bar{b}(u,\nu) + \frac{1}{\sqrt{\pi}} \sum_i \frac{\bar{\beta}_i \Gamma(\alpha_i+\frac{3}{2})}{\Gamma(\alpha_i+1)} \times \left[\frac{1}{\nu-\alpha_i} + \frac{2q^2\alpha_i}{\nu-\alpha_i+1} + \cdots + \frac{(2q^2)^{n_i}\gamma_{n_i}}{\nu-\alpha_i+n_i}\right]. \quad (7)$$

The function $\bar{b}(u,\nu)$ is regular in $\mathrm{Re}\,\nu > -\frac{1}{2}$, and the argument u of the trajectory and residue functions has been omitted. $\bar{\beta}(u)$ is the reduced Regge residue function defined by $\beta(u) = q^{2\alpha(u)} \times \bar{\beta}(u)$. Only the residues of the principal and first Khuri satellite poles have been written explicitly in (7). The significant property is that the residue of the jth satellite pole has a factor of $(2q^2)^{\alpha}$ which has poles of order up to j at $u=0$.

For $\mathrm{Re}\,\nu < N$ the analyticity of $b(u,\nu)$ at $u=0$ cannot be inferred rigorously either from the defining integral (4), which diverges, or from the Regge representation, since the latter fails to furnish the asymptotic behavior of $D_t(0,t)$. It seems impossible to avoid this difficulty, which we regard as a failure of the Regge representation rather than as a defect of the Khuri amplitudes. Therefore, we assume that the Khuri amplitudes $b(u,\nu)$ as defined by (4) can be continued to $u=0$ and have no singularities for $\mathrm{Re}\,\nu > -\frac{1}{2}$ other than those given by the finite number of moving poles in (7). Although not proven, such behavior is suggested by the maximal-analyticity concept.

Next we make a Sommerfeld-Watson transformation of the power series (5) obtaining

$$A(u,t) = (-2i)^{-1} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\nu\, (\sin\pi\nu)^{-1} b(u,\nu)(-t)^{\nu} - \sqrt{\pi} \sum_i \frac{\bar{\beta}_i \Gamma(\alpha_i+\frac{3}{2})}{\Gamma(\alpha_i+1)\sin\pi\alpha_i}[(-t)^{\alpha_i} - 2q^2\alpha_i(-t)^{\alpha_i-1} + \cdots + (-1)^n(2q^2)^{n_i}(-t)^{\alpha_i-n_i}]. \quad (8)$$

The background integral[10] defines a function with cut-plane analyticity in u and asymptotic form $0(t^{-1/2})$. Each square bracket in (8) gives the contributions of the principal and satellite Khuri poles coming from a single Regge pole, and coincides with the first n_i terms of the asymptotic series of $(2q^2)^{\alpha}P_{\alpha}(-z)$.

We consider the analyticity properties of the pole terms in Eq. (8) at $u=0$. It is shown in Ref. 8 that the reduced residues $\bar{\beta}(u)$ have no cut in the vicinity of $u=0$ but may have poles there because of the unequal-mass kinematics. The contribution of each principal Khuri pole has the same analyticity at $u=0$ as the reduced residue of the Regge pole to which it corresponds, and the jth satellite contribution has an additional singular polynomial of order j in u^{-1}. The sum of the finite number of Khuri-pole contributions must be analytic at $u=0$, and this can occur only if the singularities of the individual contributions cancel because of cooperation among the Regge trajectories.

Let $\alpha_0(u)$ be the leading Regge trajectory near $u=0$. Its reduced residue $\bar{\beta}_0(u)$ must be analytic at $u=0$, since a singularity there could not otherwise be cancelled. Then the first Khuri satellite contribution has a pole at $u=0$. To cancel this pole there must be another Regge trajectory $\alpha_1(u)$ satisfying $\alpha_1(0) = \alpha_0(0)-1$, which we call the first daughter trajectory. Its reduced residue $\bar{\beta}_1(u)$ has a pole at $u=0$, fixed so that the singular part of its principal Khuri contribution exactly cancels that of the first

Khuri satellite of the leading parent Regge pole.[11]

In general, there will be a series of daughter trajectories $\alpha_k(u)$ in the l plane satisfying

$$\alpha_k(0)=\alpha_0(0)-k, \quad k=1,\cdots,n,$$
$$\tfrac{1}{2}>\mathrm{Re}\alpha_0(0)-n>-\tfrac{1}{2}. \tag{9}$$

The corresponding reduced residues $\bar{\beta}_k(u)$ will have poles of order k at $u=0$, with everything arranged so that singularities of the individual Khuri-pole contributions cancel among themselves upon summation. Such a mechanism for the cancellation of singularities may seem miraculous, but it is a rigorous consequence of the assumed analytic behavior of $b(u,\nu)$ at $u=0$.

When the spectral function A_s is included, we find that the daughter trajectories alternate in signature, the first daughter having signature opposite to the parent. This means that the first daughter trajectory to the Pomeranchuk trajectory is unphysical at $t=0$ and does not correspond to a zero-mass scalar meson.

To obtain additional support for the daughter-trajectory hypothesis we have examined Bethe-Salpeter models, and find that the hypothesis is satisfied there for any Bethe-Salpeter kernel which Reggeizes in the first place. The invariance group of Bethe-Salpeter equations for nonzero total energy is the group O(3) of three-dimensional rotations leaving the total energy-momentum four-vector fixed. For zero total energy (i.e., $u=0$) this four-vector vanishes, and the equation becomes invariant to four-dimensional transformations of its integration variables. This extra degree of invariance at $u=0$ ensures the existence of daughter trajectories[12] (even for equal-mass kinematics) in much the same way that the extra degree of invariance which sets in as the range of a Yukawa potential becomes infinite ensures the Coulomb degeneracy of bound states. The symmetry property is independent of the ladder approximation and follows from the Lorentz invariance of general Bethe-Salpeter kernels. In Ref. 8 we show explicitly that the reduced residue of the first daughter trajectory has a pole at $u=0$ with exactly the residue necessary to cancel the singularity in the first Khuri satellite contribution of the parent Regge trajectory.

Our work suggests that each of the presently known particle trajectories is the parent trajectory to a family of daughters of the same internal quantum numbers but of alternating signature with zero-energy intercepts spaced by integers. We discuss first daughter trajectories here, which have the property that if $J^{\pm}$ is a physically realizable J^P state of the parent, then $(J-1)^{\mp}$ is a physically realizable state of the daughter.

Baryon daughters would best be detected in high-energy backward meson-baryon scattering. The Khuri representation (8) gives the correct asymptotic term to be used in fitting such experiments. The leading term $s^{\alpha(u)}$ is exactly what would come from the Legendre function of the Regge representation. However, if one wishes to include any terms of order $s^{\alpha(u)-1}$, one should include the contribution of the first daughter trajectory. A Taylor expansion about $u=0$ should be used so that cancellation of singularities there is made manifest.

The first daughter of the Pomeranchuk trajectory, $\alpha_{\mathrm{P}1}(t)$ [or the P′ daughter $\alpha_{\mathrm{P}'1}(t)$], has $B=Y=T=0$, $G=+1$, and odd signature. The ρ daughter $\alpha_{\rho 1}(t)$ has $B=Y=0$, $T=1$, $G=+1$, and even signature. Consideration of quantum numbers reveals that neither trajectory can couple to the two-body channels $\pi\pi$, $K\bar{K}$, or $N\bar{N}$, and neither would be observed in common scattering or reaction processes. These trajectories do couple to unequal-mass channels and could, in principle, be observed in double production processes such as $N+N\rightarrow N_{1/2}{}^*+N_{1/2}{}^*$ in which two $T=\frac{1}{2}$ nucleon isobars are produced. The daughter trajectories would be necessary in such processes to resolve kinematic ambiguities in the Regge representation similar to those for backward scattering.

It is difficult theoretically to predict the behavior of daughter trajectories away from zero energy, but it is tempting to consider the possibility that they are roughly parallel with the parents. If so, there would be a physical vector meson of mass between 1.2 and 1.6 BeV on the P1 trajectory, and a scalar meson of mass between 700 and 1100 MeV on the $\rho 1$. Neither could decay into two pseudoscalar mesons. The 1^- Pomeranchuk daughter trajectory could decay into $K\bar{K}\pi$ with p-wave angular-momentum barriers in the configuration $K^*\bar{K}$, or d-wave barriers in the configuration $(K\bar{K})\pi$, but the quantum numbers prohibit the 0^+ ρ daughter from decaying into $K\bar{K}\pi$. Both particles have 4π decay modes, the 1^- into the configuration $\rho\rho$ and the 0^+ into the configuration $\sigma\rho$. Both particles can decay electromagnetically to $\pi\pi\gamma$, and this mode may be dominant for the 0^+. The present experimental situation, although not conclusive, does not seem favorable to the existence of these mesons. This would indicate that the daughter trajectories have slopes more shallow than the parents.

We have used the Khuri representation to characterize the behavior of partial-wave amplitudes $a(u,l)$ at $u=0$ and find

$$a^{\pm}(u,l) \approx \frac{\bar{\beta}^{\pm}(0)}{\pi^{3/2}} \frac{\Gamma[\alpha^{\pm}(0)+\frac{3}{2}]}{\Gamma[\alpha^{\pm}(0)+l+2]} \left[\frac{(m^2-\mu^2)^2}{u}\right]^{\alpha^{\pm}(0)}, \quad (10)$$

where $\alpha^{\pm}(0)$ is the zero-energy intercept of the leading Regge trajectory of the same signature in the direct channel, and $\bar{\beta}^{\pm}(0)$ is its reduced residue. The behavior (10) applies if $\alpha^{\pm}(0)>-1$, otherwise $a(u,l)\sim u\log u$. The proof involves a straightforward estimate of the integrals in the Froissart-Gribov definition of $a(u,l)$ and is given in Ref. 8. It is not surprising that it is the high-energy behavior in crossed channels which determines the behavior of partial-wave amplitudes at $u=0$ in the unequal-mass case, since the integral from $z=-1$ to $z=+1$ which defines (physical) partial-wave amplitudes corresponds to an integration of infinite range in the Mandelstam variables at $u=0$. The behavior (10) is in contradiction to what has generally been believed[7,13] and may very well have interesting implications for dynamical calculations. We expect that (10) characterizes the behavior of $a(u,l)$ for $|u| \ll (m^2-\mu^2)$.

Goldberger and Jones[14] have recently written a paper in which the same subject is approached from a different point of view. Different results are found largely because these authors fail to take into account the mechanism of cancellation of singularities by daughter trajectories. They find that the condition $\alpha(0)<\frac{1}{2}$ must be satisfied for the consistency of their method. This condition would seem to be violated by the Pomeranchuk trajectory which certainly couples to unequal-mass channels and in Bethe-Salpeter models which have all the analyticity properties used by Goldberger and Jones. Since the daughter trajectory hypothesis is definitely satisfied in Bethe-Salpeter models, we feel that it is the correct mechanism by which the ambiguity in the Regge representation is resolved.

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[1]W. R. Frisken et al., Phys. Rev. Letters 15, 313 (1965).

[2]C. T. Coffin et al., Phys. Rev. Letters 15, 838 (1965).

[3]G. F. Chew and J. D. Stack, University of California Radiation Laboratory Report No. UCRL-16293 (unpublished); J. D. Stack, Phys. Rev. Letters 16, 268 (1965).

[4]V. Barger and D. Cline, Phys. Rev. Letters 16, 913 (1966).

[5]For example, see M. Gell-Mann, S. C. Frautschi, and F. Zachariasen, Phys. Rev. 126, 2204 (1962), Ref. 15.

[6]D. A. Atkinson and V. Barger, Nuovo Cimento 38, 634 (1965).

[7]S. C. Frautschi and J. D. Walecka, Phys. Rev. 120, 1486 (1960).

[8]D. Z. Freedman and J. M. Wang, to be published.

[9]N. N. Khuri, Phys. Rev. 132, 914 (1963).

[10]In Ref. 8 we show that the Khuri representation can be extended to the left of $-\frac{1}{2}$ as far as the corresponding Regge representation (in Mandelstam's form) can be extended.

[11]This possibility was first suggested by Professor S. Mandelstam.

[12]G. Domokos and P. Suranyi, Nucl. Phys. 54, 529 (1964).

[13]E. S. Abers and V. L. Teplitz derived a more primitive form of the result (10) by using the Froissart bound as input [Nuovo Cimento 39, 739 (1965)].

[14]M. L. Goldberger and C. E. Jones, to be published.