EFFECT OF HALL CURRENT ON RAYLEIGH-TAYLOR INSTABILITY OF A PLASMA

N. K. Nayyar* and S. K. Trehan*
Laboratory for Theoretical Studies, Goddard Space Flight Center, Greenbelt, Maryland
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It has been recently pointed out by Hosking[1] that the inclusion of Hall current in Ohm's law produces a new aperiodic instability in a wave number band which was previously stable in the classical problem of Rayleigh-Taylor instability for a plasma.[2,3] However, it will be shown here that this result is incorrect and is a consequence of the fact that Hosking has not solved the problem with due regard to the boundary conditions. It is the aim of this note to show that the inclusion of Hall-current and electron-inertia terms in the generalized Ohm's law, in fact, do not have any effect on the development of Rayleigh-Taylor instability in hydromagnetics.

Consider then a situation where an infinitely conducting plasma occupies the half-space $0<z<\infty$ and is supported against gravity (acting along the negative z axis) by a uniform magnetic field which we shall take along the x axis of a system of Cartesian coordinates x, y, z. We shall assume that the medium is incompressible. The equations basic to our discussions are the equation of motion

$$\rho\frac{d\vec{v}}{dt}=-\vec{\nabla}p+\frac{1}{4\pi}(\vec{\nabla}\times\vec{B})\times\vec{B}+\sigma\vec{E}+\rho\vec{g} \tag{1}$$

and the generalized Ohm's law

$$\vec{E}+\frac{1}{c}\vec{v}\times\vec{B}-\frac{1}{4\pi N_e e}(\vec{\nabla}\times\vec{B})\times\vec{B}+\frac{1}{N_e e}\vec{\nabla}p_e=0 \tag{2}$$

together with

$$\vec{\nabla}\cdot\vec{v}=0 \text{ and } \vec{\nabla}\cdot\vec{B}=0, \tag{3}$$

where $\vec{g}=(0, 0, -g)$ is the constant gravitational field, σ is the charge density, N_e is the electron number density, p_e is the electron pressure, p is the total fluid pressure, $\vec{E}$ is the electric field, ρ is the constant density, $\vec{v}$ is the fluid velocity, $\vec{B}$ is the magnetic field, and c is the velocity of light. The electric and magnetic fields satisfy Maxwell's equations in which we shall neglect the displacement currents. We shall assume equal electron and ion pressures so that we can write $p_e=p_i=p/2$. Thus in equilibrium Eqs. (1) and (2) give $\vec{\nabla}p=\rho\vec{g}$ and $\vec{E}=-\vec{\nabla}p/2N_e e=\vec{e}_z\rho g/2N_e e$. Since we assume charge neutrality in equilibrium, the density ρ is equal to $N(M+m)\approx NM$, where M and m are the masses of the ion and electron, respectively. Thus $\vec{E}=\vec{e}_z gB_p/2c\omega_i$ where ω_i is the ion gyrofrequency and B_p represents the magnetic field inside the plasma. In equilibrium the condition of pressure balance at the surface obtained by integrating the equation of motion (1) is

$$p+\frac{B_p^2}{8\pi}+\frac{E_v^2}{8\pi}=\frac{B_v^2}{8\pi}+\frac{E_p^2}{8\pi}, \tag{4}$$

where the subscripts p and v represent quantities in plasma and vacuum, respectively. Hosking has taken the magnetic field to be uniform throughout space ($B_p=B_v$ and $E_v=0$), so that the plasma pressure at the surface becomes $p=E_p^2/8\pi=g^2(B_p^2/8\pi)/(4c^2\omega_i^2)$. It can be readily seen that for any situation of physical interest, this is not a realizable model. For if we take a typical situation where $B_p\sim 10^4$ G, $g\sim 10^3$ cm/sec^2 for a hydrogen plasma, $(g^2/4c^2\omega_i^2)(B_p^2/8\pi)\simeq 10^{-25}$. This gives $N\sim 10^{-15}/$cm^3 for a plasma with $T\sim 10^6$ °K. In fact, for $p\neq 0$, the condition of pressure balance requires a jump in the magnetic field at the interface $z=0$ and one can ignore the electric field in this condition. The condition for pressure balance then becomes simply $p+B_p^2/8\pi=B_v^2/8\pi$. This is compatible with the remark by Spitzer[4] that in those situations where an electric field may exist in a plasma which is initally electrically neutral, deviation from charge neutrality must be taken into account in the Poisson's equation while these may be ignored in the equation of motion. This shall be done henceforth.

The linearized equations governing the departures from equilibrium in the plasma are

$$\rho\frac{\partial\vec{v}}{\partial t}=-\vec{\nabla}(p_1+\vec{B}_p\cdot\vec{b}_p/4\pi)+\frac{1}{4\pi}(\vec{B}_p\cdot\vec{\nabla})\vec{b}_p, \tag{5}$$

and the induction equation

$$\frac{\partial\vec{b}_p}{\partial t}=\vec{\nabla}\times(\vec{v}\times\mathrm{B}_p)-c_1\vec{\nabla}\times[(\vec{\nabla}\times\vec{b}_p)\times\vec{B}_p], \tag{6}$$

where $c_1=c/4\pi N_e e=B_p/4\pi\rho\omega_i$, and $\vec{b}_p$, p_1, and $\vec{v}$ are the perturbations in the magnetic field pressure, and velocity, respectively. The ∇p term drops out when we take the curl of Eq. (2), as N is constant in equilibrium. We now assume that all the perturbed quantities have the form $f(x,y,z,t)=f(z)\exp(nt+ik_x x+ik_y y)$. Equations (5) and (6) then reduce to

$$\rho n\vec{v}=-\vec{\nabla}\Pi_1+\frac{ik_x}{4\pi}B_p\vec{b}_p,\quad (\Pi_1=p_1+\vec{B}_p\cdot\vec{b}_p/4\pi), \tag{7}$$

and

$$n\vec{b}_p=ik_xB_p\vec{v}-ic_1k_xB_p\vec{\nabla}\times\vec{b}_p. \tag{8}$$

It is interesting to observe from Eqs. (7) and (8) that if $k_x=0$, i.e., the perturbation is normal to both $\vec{g}$ and $\vec{B}_p$, then $\vec{b}_p=0$. Thus we conclude that inclusion of Hall current does not affect the development of instability in case the perturbation is perpendicular to the direction of the magnetic field.

We shall now consider the case when $k_y=0$ and $k_x\neq 0$. We shall also drop the subscript x on k_x. This is the case considered by Hosking. From Eqs. (7) and (8) we obtain

$$\vec{\nabla}\times\left[\frac{i\Omega_A^{\,2}}{kn\omega_i}\vec{\nabla}\times\vec{b}_p+\left(1+\frac{\Omega_A^{\,2}}{n^2}\right)\vec{b}_p\right]=0, \tag{9}$$

where $\Omega_A^{\,2}=k^2B_p^{\,2}/4\pi\rho$. This equation can be intergrated to give

$$\frac{i\Omega_A^{\,2}}{kn\omega_i}\vec{\nabla}\times\vec{b}_p+\left(1+\frac{\Omega_A^{\,2}}{n^2}\right)\vec{b}_p=\vec{\nabla}\varphi, \tag{10}$$

where φ is some scalar function. Making use of the solenoidal character of $\vec{b}_p$, it follows from Eq. (10) that $\nabla^2\varphi=0$. The solution of this equation which is valid in the upper-half plane $z\geqslant 0$ is clearly $\varphi=\varphi_0\exp(-kz)$ where φ_0 is a constant. On substituting this into Eq. (10), we obtain after some straightforward reductions the following components of the perturbed magnetic field:

$$b_x=-(i\alpha/k)\varphi_1e^{-\alpha z}-(ik/\lambda)\varphi_0e^{-kz}, \tag{11}$$

$$b_y=-(i\lambda/k)\varphi_1e^{-\alpha z}, \tag{12}$$

$$b_z=\varphi_1e^{-\alpha z}+(k/\lambda)\varphi_0e^{-kz}, \tag{13}$$

where $\alpha^2=k^2-\lambda^2$, φ_1 is a constant, and

$$\lambda^2=-\left(\frac{kn\omega_i}{\Omega_A^{\,2}}\right)^2\left(1+\frac{\Omega_A^{\,2}}{n^2}\right)^2. \tag{14}$$

From Eq. (7) we find that $\nabla^2\Pi_1=0$ and, therefore, $\Pi_1=\varphi_2\exp(-kz)$. Eliminating $\vec{v}$ from Eqs. (7) and (8) and making use of Eqs. (11)-(13), we find that

$$\varphi_2=-\frac{i\rho n^2}{k\lambda B_p}\left(1+\frac{\Omega_A^{\,2}}{n^2}\right).$$

In the vacuum the magnetic field is clearly given by

$$\vec{b}_v=[ik\vec{e}_x+k\vec{e}_z]\varphi_3e^{kz},\quad z\leqslant 0, \tag{15}$$

where φ_3 is another constant, and in writing Eqs. (11)-(15), we have suppressed the common phase factor $\exp(nt+ikx)$.

The dispersion relation will now be obtained by matching the boundary conditions which are as follows: (1) The z component of the velocity at $z=0$ must be compatible with the assumed form of the deformed boundary $z=\epsilon\exp(nt+ikx)$; (2) the normal component of the magnetic field; and (3) the normal stress tensor must be continuous at the deformed boundary. As the equilibrium magnetic field is discontinuous at $z=0$, it can be shown that the normal component of the total magnetic field must vanish at the deformed boundary. With the assumed form of dependence on x, the normal to the perturbed surface is given by $\vec{e}_n=\vec{e}_z-ik\epsilon\vec{e}_x$. Remembering that at the surface $v_z=n\epsilon$ and making use of the first boundary condition, Eq. (8) gives

$$\varphi_1\left(n+\frac{i\lambda}{k}\frac{\Omega_A^{\,2}}{\omega_i}\right)+\frac{nk}{\lambda}\varphi_0-ikn\epsilon B_p=0. \tag{16}$$

The vanishing of the normal component of the magnetic field at the deformed boundary leads to

$$\varphi_1+\frac{k}{\lambda}\varphi_0-ikB_p\epsilon=0 \tag{17}$$

and

$$k\varphi_2-ikB_v\epsilon=0. \tag{18}$$

From Eqs. (16) and (17) we get $\varphi_1=0$. Finally, we have to satisfy the condition of the continuity of the stress tensor across the deformed boundary. This requires that

$$\epsilon\frac{dp}{dz}+p_1+\frac{B_pb_{px}}{4\pi}=\frac{B_vb_{vx}}{4\pi}. \tag{19}$$

On making use of the foregoing relations and after straightforward reductions this leads to the dispersion relation

$$n^2 = gk_x - (kx^2/4\pi\rho)(B_p^{\,2}+B_v^{\,2}), \qquad (20)$$

where we have now restored the subscript on k. In the general case where k_x and k_y are both present, Eq. (20) is replaced by

$$n^2 = g(kx^2+ky^2)^{1/2} - (kx^2/4\pi\rho)(B_p^{\,2}+B_v^{\,2}), \qquad (21)$$

which is identical with the expression obtained in the classical case using the simple form of Ohm's law $\vec{E}+\vec{v}\times\vec{B}/c=0$.

It is thus clear that the inclusion of the Hall current does not have any effect on the stability of a plasma supported against gravity.

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*On leave of absence from the Department of Physics and Astrophysics, University of Delhi, Delhi, India. Resident Research Associates of the National Academy of Sciences, National Research Council.

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TEMPERATURE MEASUREMENTS OF A LASER SPARK FROM SOFT-X-RAY EMISSION*

A. J. Alcock, P. P. Pashinin,† and S. A. Ramsden
Division of Pure Physics, National Research Council, Ottawa, Canada
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Previous studies of laser sparks have shown that most of the energy of the beam is absorbed in the spark creating a high-density plasma,[1,2] and that during the laser pulse the ionized region expands[3-5] at high velocity into the beam.

On the basis of the high absorption of energy in a small volume and also from the dynamics of the spark, it was expected that a high temperature would be produced during the laser pulse. An estimate of the electron temperature has been made by Mandelstam et al.[4] from the absolute value of the soft-x-ray emission detected with a Geiger-Muller counter. The method involved assumptions about the size of the radiating volume, the existence of thermal equilibrium, and the duration of x-ray emission. To eliminate these uncertainties we have made temperature measurements of laser sparks from the relative intensity of the x-ray flux transmitted through beryllium foils of different thickness.[6] The x-ray emission was detected by means of two plastic scintillators (NE 102) each of which was optically coupled to a photomultiplier (Philips 56 AVP). The fast response of such a detection system permitted a much better time resolution than can be obtained using a Geiger counter.

The experimental arrangement used to investigate various gases over a range of pressures is shown in Fig. 1. The output beam from a 300-400 MW, Q-spoiled, ruby laser was focused inside a pressure cell by a lens whose focal

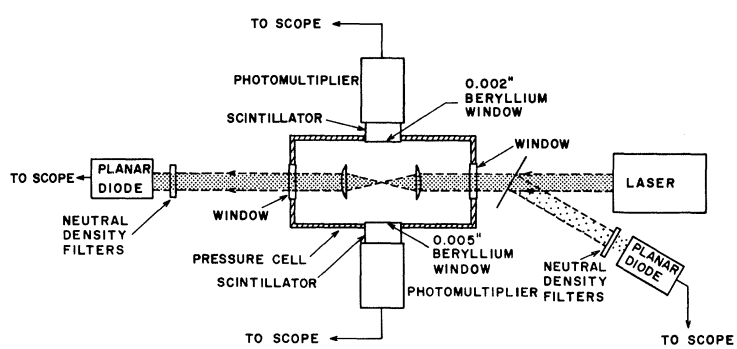

FIG. 1. Diagram of apparatus used to detect soft-x-ray emission from a laser spark.