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SINGLE-PARTICLE WAVE FUNCTIONS OF CALCIUM ISOTOPES

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The recent very accurate measurements[1-3] of the differential cross section for elastic electron scattering by Ca^{40}, Ca^{44}, and Ca^{48}, together with our knowledge of single-particle energies for both protons and neutrons,[4-11] particularly for Ca^{40} and Ca^{48}, have made it possible to determine with considerable certainty shell-model potentials of Woods-Saxon form which will generate single-particle wave functions to fit the data.

Our procedure has been to generate single-particle wave functions in an energy-dependent Woods-Saxon well,

$$V(r)=-V_E f(r)-V_{SE}\frac{1}{r}\frac{d}{dr}f(r)\vec{1}\cdot\vec{\sigma}\left(\frac{\hbar}{m_\pi c}\right)^2+V_C,$$

where the strengths of the central and spin-orbit parts, V_E and V_{SE}, are different for the different shells, V_C is the Coulomb potential for the protons, and

$$f(r)=\left[1+\exp\left(\frac{r-R}{a}\right)\right]^{-1},\quad R=r_0(A-1)^{1/3}.$$

The parameters are adjusted so that the binding energies are fitted to the separation energies obtained from $(p,2p)$[4] and (e,ep)[5] reactions, as well as from mass differences, and the charge density of the protons, as obtained from the wave functions after folding in the finite proton size, is fitted to elastic electron scattering.[1-3] The need for an energy-dependent well follows from the very deep binding of 1s-shell protons in nuclei between oxygen and calcium, as found from the (e,ep) reaction.[5,12]

The description of the nuclear wave function in terms of a product of single-particle wave functions is most likely to be successful for nuclei with closed shells and for that reason we undertook an investigation of the calcium

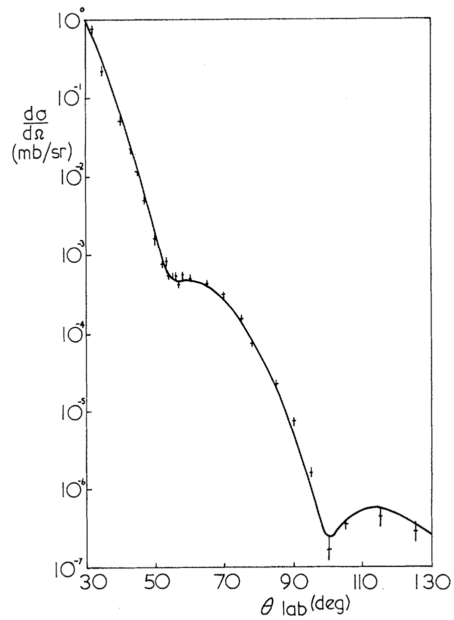

FIG. 1. Elastic electron scattering by Ca^{40} at 250 MeV.

isotopes. We first fitted the data for Ca^{40} (see Fig. 1) and then adjusted the parameters r_0 and a, as well as the depths of the potentials for the $2s$-$1d$ shell in order to fit the data for Ca^{44} (see Fig. 2) and Ca^{48} (see Fig. 3). It was found that, for Ca^{44}, r_0 and a could be left unaltered, but that for Ca^{48} both had to be decreased significantly. The spin-orbit potential was left unchanged, while the change in the depth of the central potential was 0.5 MeV for each excess neutron. This leads to a symmetry term in the potential of strength $U_1 \simeq 80$ MeV, which is close to the value obtained by other methods.[13] The parameters for all the proton potentials and comparisons between calculated and experimental binding energies are given in Table I. It will be seen that the neutron potential for the $2s$-$1d$ shell in Ca^{40}, which is obtained by simply omitting the Coulomb part from the proton potential, leads to single-particle neutron levels in very good agreement with results from the (p,d) reaction.[8] We also estimate parameters for the neutron potential of the $2p$-$1f$ shell in Ca^{40} to agree with the results from the (d,p) reaction,[9] and then, using again $U_1 \simeq 80$ MeV, obtain values for the separation energies[6,7,10] in the $2p$-$1f$ shell of neutrons in Ca^{48} and of protons in both Ca^{40} and Ca^{48}. There are, of course, no data from electron scattering for these, but the proton potential for Ca^{48} turns out to be very similar to one used recently elsewhere.[14] It will be seen from Table I that agreement with experiment is fairly good for the $1f$ levels, although a slight alteration in the potential parameters may be required, if account is taken of the self-binding energy[15] of the $1f_{7/2}$ neutrons in Ca^{48}.

The $2p$ binding energies are consistently too small, which is in agreement with other observations[15-17] that the effective mass of a nucleon is considerably larger for energies just above the Fermi surface than for bound states. Finally, the results of the rms radii of the charge distributions of Ca^{40} and Ca^{44} are in good agreement with the results from muonic x rays.[18]

Although the number of parameters appears to be large, it is, in fact, not at all easy to fit all the data, so that for instance it is quite impossible to obtain a fit for the Ca^{48} data with $r_0 = 1.30$ and $a = 0.65$. Estimated errors for V_E in the $2s$-$1d$ shell, and for r_0 and a are better than 2, 1, and 10%, respectively. The resulting charge distributions are shown in Fig. 4. These are in extremely close agreement with best fits for charge distributions of the Fermi and modified Gaussian types,[1-3] except that there is an increase in the central charge density. This is due to the $2s$ protons, and although it is almost certainly a real effect, it is not one that is sensitive to electron scattering. Since it results, however, in a different value for the rms radius, it should be sensitive to accurate measurements of muonic x-ray energies. At present, both charge distributions give results within the experimental error.

It will be seen that there is essentially no difference between the potential parameters for Ca^{40} and Ca^{44} and that the well actually increases in size according to an $A^{1/3}$ proportionality. That the rms radius does not increase

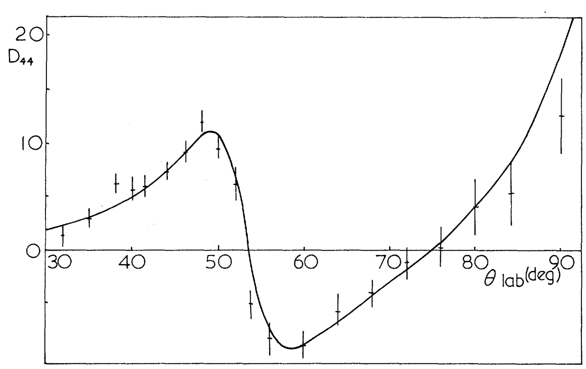

FIG. 2. Elastic electron scattering by Ca^{44} and Ca^{40} at 250 MeV. $D_{44} = 100(\sigma_{40} - \sigma_{44})/(\sigma_{40} + \sigma_{44})$.

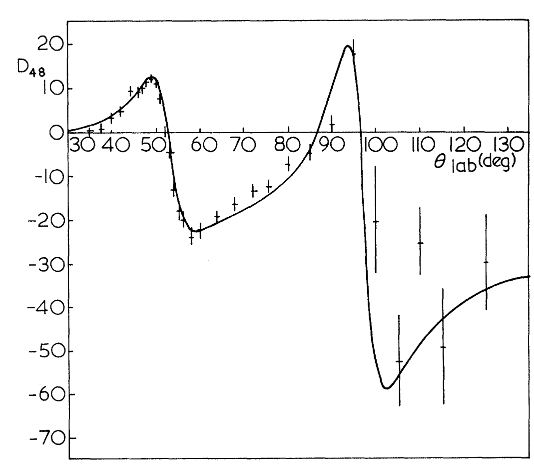

FIG. 3. Elastic electron scattering by Ca^{48} and Ca^{40} at 250 MeV. $D_{48} = 100(\sigma_{40} - \sigma_{48})/(\sigma_{40} + \sigma_{48})$.

Table I. Well parameters and energy levels of $Ca^{40,44,48}$. (Distances in F, energies in MeV.)

		Ca^{40} $r_0=1.30$, $a=0.60$, $\langle r^2\rangle^{1/2}=3.39$					Ca^{44} $r_0=1.30$, $a=0.60$, $\langle r^2\rangle^{1/2}=3.42$					Ca^{48} $r_0=1.26$, $a=0.45$, $\langle r^2\rangle^{1/2}=3.38$				
Nucleon	Level	V_E	V_{SE}	$E_{\text{theor.}}$	$E_{\text{expt.}}$	Reaction	V_E	V_{SE}	$E_{\text{theor.}}$	$E_{\text{expt.}}$	Reaction	V_E	V_{SE}	$E_{\text{theor.}}$	$E_{\text{expt.}}$	Reaction
Protons	$1s_{1/2}$	85	⋯	62.9	$\sim 65^a$	(e,ep)	85	⋯	63.9			85	⋯	65.6		
	$1p_{3/2}$	60	30	32.1			60	30	33.2			60	30	34.8		
	$1p_{1/2}$	60	30	24.5	24.5^b	$(p,2p)$	60	30	26.3			60	30	28.8		
	$1d_{5/2}$	53	12	15.2	15.1^b	$(p,2p)$	55	12	18.4			57	12	21.8		
	$2s_{1/2}$	53	12	10.1	10.9^b	$(p,2p)$	55	12	13.1			57	12	15.5		
	$1d_{3/2}$	53	12	8.5	$8.3^{b,c}$	$(p,2p),B(p)$	55	12	12.0	12.2^c	$B(p)$	57	12	15.5	15.3^c	$B(p)$
	$1f_{7/2}$	50	8.3	1.7	1.1^c	$B(p)$						54	8.3	8.0	9.6^c	$B(p)$
	$2p_{3/2}$	50	8.3	⋯	-0.7^d	(He^3,d)						54	8.3	1.9	6.1^e	(He^3,d)
	$2p_{1/2}$	50	8.3	⋯								54	8.3	⋯	3.7^e	(He^3,d)
	$1f_{5/2}$	50	8.3	⋯								54	8.3	1.2	4.9^e	(He^3,d)
	$1d_{5/2}$	53	12	22.6	21.9^f	(p,d)										
	$2s_{1/2}$	53	12	17.6	18.2^f	(p,d)										
	$1d_{3/2}$	53	12	16.0	15.6^c	$B(n)$										
Neutrons	$1f_{7/2}$	50	8.3	8.5	8.4^c	$B(n)$						46	8.3	9.5	9.9^c	$B(n)$
	$2p_{3/2}$	50	8.3	4.9	6.3^g	(d,p)						46	8.3	4.0	5.1^c	$B(n)$
	$2p_{1/2}$	50	8.3	3.1	4.3^g	(d,p)						46	8.3	1.8	3.1^h	(d,p)
	$1f_{5/2}$	50	8.3	2.3	2.9^g	(d,p)						46	8.3	1.9	1.2^h	(d,p)

[a]Ref. 5. [b]Ref. 4. [c]Ref. 11 [d]Ref. 7. [e]Ref. 6. [f]Ref. 8. [g]Ref. 9. [h]Ref. 10

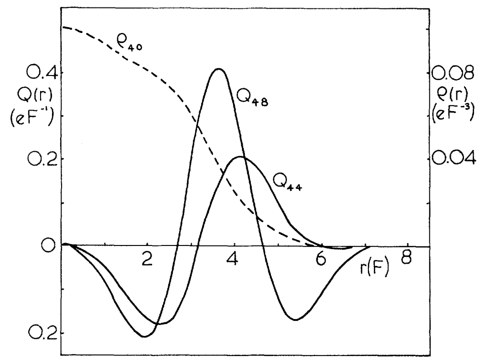

FIG. 4. Charge density $\rho(r)$ for Ca^{40} (right-hand scale) and difference between charge densities of $Ca^{44,48}$ and Ca^{40} (left-hand scale). $Q_{44,48} = 4\pi r^2(\rho_{44,48} - \rho_{40})$.

correspondingly is due to the greater binding of the last protons in Ca^{44}, which leads to a drawing in of the wave functions in the sloping well. There appear to be no special shell closure effects in Ca^{40}, and this is presumably because the very existence of a stable calcium isotope with $N = Z$ is due to its doubly magic nature, and no increased stability over neighboring nuclides is to be expected. The situation is very different for Ca^{48}, which is of course also doubly magic. Certainly, the greater binding of the last protons is by itself unable to account for the observed facts here, and the very much smaller values obtained for r_0 and a are a likely consequence of the closed-shell structure and great stability of the ground state of Ca^{48}. This confirms the observations[9,19] that the ground state of Ca^{40} may well have a more complicated configuration than a simple doubly closed-shell configuration and that the excitation of collective states is considerably stronger in Ca^{40} than in Ca^{48}. It is noteworthy too that the value of a for Ca^{48} is at least 20% smaller than that obtained for any other nucleus—including O^{16}—that has been investigated in the manner described in this paper.[20]

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