SINGULARITIES IN THE UNIVERSE

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It is shown that space-time cannot be singularity free if the Einstein equations hold, matter has normal properties, and some reasonable global conditions hold. This would suggest that the Einstein theory probably breaks down in very strong fields.

The prediction of a singularity by a physical theory is usually taken as an indication that the theory has broken down. Thus the question of whether Einstein's theory of general relativity predicts singularities of space-time in all reasonable cosmological models is clearly of considerable importance.

It has been known for some time that certain exact solutions of the Einstein equation have singularities[1-5] but it was claimed[6] that this might simply be a consequence of the high symmetry of these solutions and that the singularities might be avoided by making small perturbations. This was shown not to be the case by Penrose[7] who proved that an approximately spherical star collapsing in a universe with a noncompact Cauchy surface must go to a singularity. Penrose's method can also be applied to prove that an approximately homogeneous and isotropic "open" cosmological model must have a singularity in the past.[8] However the method can only be applied to models with noncompact Cauchy surfaces and it might be that the universe has a compact Cauchy surface (a "closed" model) or even no Cauchy surface at all. The aim of this paper is to indicate that, on certain reasonable assumptions, it is still possible to prove that singularities must occur. Only the results and a brief indication of the method will be given here. The full details of the proofs are being published elsewhere.

We assume the Einstein equations: $R_{ab}-\frac{1}{2}g_{ab}\times R=T_{ab}$. Space-time is said to be singularity free if it is time-complete[9] (all timelike geodesics can be extended to arbitary length) and if the metric is a C^2 tensor field.

Theorem 1.—Space-time cannot be singularity free if (1) the energy-momentum tensor obeys the inequality $T_{ab}w^aw^b \geqslant \frac{1}{2}w_aw^aT$ for any timelike or null vector w^a (for a fluid with density μ and isotropic pressure p this is satisfied if $\mu+p \geqslant 0$ and $\mu+3p \geqslant 0$; this is a very reasonable requirement); (2) there exists a Cauchy surface H [we define a Cauchy surface as a closed (not necessarily compact) spacelike surface without boundary, which intersects every timelike and null line once and once only]; and (3) the unit normals v^a to H have a positive lower bound of divergence on H; i.e., $v^a{}_{;a} \geqslant C > 0$.

If condition (2) were satisfied one would expect condition (3) to be satisfied in a universe expanding everywhere. Indeed it might almost serve as a definition of what we mean by an expanding universe. Nevertheless, it would be impossible to test it by observation. We therefore present two theorems which replace this condition by other conditions. A further theorem is given in the note by Geroch accompaning this paper.[10]

Theorem 2(a).—Spacetime is not singularity free if conditions (1) and (2) hold and if (4) there is a point p on H such that all the past-directed timelike geodesics through p start converging again. That is, $\theta=\mu^\alpha{}_{;\alpha}$ becomes negative on each past-directed timelike geodesic through p, where μ^a is the unit tangent vector to the geodesic.

Condition (4) is fairly severe but it will be satisfied in an approximately homogenous and isotropic expanding universe obeying (1) and (2). It would be testable by observation.

Theorem 2(b).—Space-time is not singularity free if condition (1) holds, there is a compact Cauchy surface H, and (5) on H the energy-momentum tensor satisfies the strict inequality

$$T_{ab}w^aw^b > \frac{1}{2}w_aw^aT$$

for any timelike or null vector w^a.

Condition (5) requires that the density is nonzero on H. It is assumed for simplicity but is really stronger than is necessary. All that is needed is that every timelike geodesic should encounter some matter or even some randomly oriented curvature.

Penrose[11] has pointed out that the universe might not possess a Cauchy surface at all. He instances the Riesner-Nordstrom[12] solution as an example of a space-time where it is impossible to find a connected spacelike surface cutting every timelike line. We will call a connected closed spacelike surface without boundary a slice and a slice which does not intersect any timelike or null line more than once a partial Cauchy surface. It is easy to show that if space-time admits a slice then either that slice is a partial Cauchy surface or space-time has a covering space in which each connected component of the image of the slice is a partial Cauchy surface. We may apply the following theorem to space-time or to the covering space since a singularity in the covering space implies one in the space covered.

Theorem 3. – Space time is not singularity free if condition (1) holds and (6) there exists a compact partial Cauchy surface H whose unit normals v^a are diverging, (7) every point q has a neighborhood W such that every timelike and null line from q leaves W and does not re-enter.

Condition (7) is really a statement about causality.[13] It would seem a very reasonable requirement.

Finally, we make a brief mention of the technique used to prove these results. A point p is said to be conjugate to a point q along a timelike geodesic γ if p lies on the caustic of timelike geodesics through q. The point p is said to be conjugate to a spacelike surface H if it lies on the caustic of the geodesics normal to H. By the formula for variation of arc length it can be shown that a geodesic γ normal to H from H to a point r cannot be the longest timelike line from H to r if there is a point conjugate to H on γ between H and r. By the conditions of the theorems it is possible to show that there is a point conjugate to H within a bounded distance b on every geodesic normal to H. On the other hand, it is possible to show that there are points from which there is a longest timelike line to H of finite length greater than b. This establishes a contradiction which shows that the assumed conditions are incompatible with space-time being singularity free.

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SINGULARITIES IN CLOSED UNIVERSES

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In recent years, several theorems[1,2] have appeared which show that the singularities observed in certain exact solutions of the Einstein equations cannot be avoided by small perturbations in these solutions. These theorems are not sufficiently strong, however, to settle completely the question whether any nonsingular solutions of Einstein's equations can provide a reasonable model of our own universe. In particular, should the trapped surface of Penrose[1] or the expanding Cauchy surface of Hawking[2] occur in our own immediate vicinity, we could, by suitably distributing masses and thus affecting the local geometry, destroy these symptons of prospective collapse. Would we thus save the entire universe from evolving into a singular state? This seems unlikely, and one would like to express this feeling as a theorem.

We should like here to present a result which takes a further step toward the more limited goal of establishing that, with some suitable