A. Kernan, W. M. Powell, U. Camerini, D. Cline, W. F. Fry, J. G. Gaidos, D. Murphree, and C. J. Murphy, Phys. Rev. 139, B1600 (1965).

[12]This is obtained from Table I and Eq. (12) of N. Cabibbo and A. Maksymowicz, Phys. Rev. 137, B438 (1965), by setting $a_0=0$ and correcting an error of a factor of 4. [I wish to thank Dr. G. Kalmus for pointing out this mistake; the formula is given correctly in Eq. (9) of Ref. 11.] An a_0 equal to one or more pion Compton wavelengths would radically change Eq. (26).

[13]The pole comes from diagrams in which the K meson emits two soft pions, continues as a K, and then decays into $l+\nu$. The $K\to l+\nu$ amplitude is proportional to $p_K^{\lambda}=(k-q-p)^{\lambda}$, to this pole appears only in F_3. Its residue can be determined from the predicted $KK\pi\pi$ amplitude (see Ref. 14), with a result in complete agreement with Eqs. (19), (20), and (13).

[14]Y. Tomozawa, to be published; and Ref. 3.

REPRESENTATION OF LOCAL CURRENT ALGEBRA AT INFINITE MOMENTUM*

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It has been proposed[1,2] that the charge operators F_i and F_i^5 of the vector and axial-vector current octets may obey the algebra of SU(3) $\times$SU(3) exactly, to all orders in the strong interactions, under equal-time commutation. The charge-density operators $\mathfrak{F}_{i0}(\vec{x})$ and $\mathfrak{F}_{i0}{}^5(\vec{x})$ will generate a local infinite-parameter algebra, the direct sum of an infinite number of SU(3)$\times$SU(3) algebras, if we make the additional assumption[1,2] that the equal-time commutators of these densities contain only a spatial δ function and not gradients thereof. The Fourier components $F_i(\vec{k})$ and $F_i^5(\vec{k})$ then obey the commutation rules

$$[F_i(\vec{k}),F_j(\vec{k}')]=if_{ijk}F_k(\vec{k}+\vec{k}'),$$
$$[F_i(\vec{k}),F_j{}^5(\vec{k}')]=if_{ijk}F_k{}^5(\vec{k}+\vec{k}'),$$
$$[F_i{}^5(\vec{k}),F_j{}^5(\vec{k}')]=if_{ijk}F_k(\vec{k}+\vec{k}'). \qquad (1)$$

It has also been suggested[3-5] that limited numbers of stable and resonant particle states may form simple approximate representations of the single SU(3)$\otimes$SU(3) generated by the operators $F_i(0)$ and $F_i^5(0)$, thereby allowing one to classify states according to the algebra and to predict the matrix elements of the operators. We would now like to suggest that the full algebra be used in a similar spirit, leading to the classification of infinite numbers of states according to (approximate) representations of (1) and predictions for complete form factors. The purpose of this Letter is to give a specific, concrete formulation of this program.

The most effective work on the commutation relations has been done following the suggestion of Fubini and Furlan[6] that they be sandwiched between states with infinite momentum, say $p_z=\infty$. This idea, which is equivalent to using dispersion relations in the s variable, was utilized by Adler[7] and Weisberger[8] in their formulation of the sum rule for axial-vector coupling constants. The procedure can be generalized to the algebra of charge densities (1), using Fourier components $\vec{k}_\perp$ perpendicular to the z axis. Adler[9] treated the case $\vec{k}_\perp=-\vec{k}_\perp'$; the case of general $\vec{k}_\perp$ and $\vec{k}_\perp'$ has been discussed by Fubini[10] and by the authors,[5] who reviewed several useful features of taking $p_z=\infty$.

Here we would like to point out an additional important property of the limit $p_z\to\infty$, namely that the matrix element of $\mathfrak{F}_{0i}(\vec{x})$ or $\mathfrak{F}_{0i}{}^5(\vec{x})$ between a state with transverse momentum $\vec{p}_\perp$ and $p_z=\infty$ and another with transverse momentum $\vec{p}_\perp'$ and $p_z=\infty$ does not depend[11] on the sum $\vec{p}_\perp'+\vec{p}_\perp$ but only on the difference $\vec{k}_\perp=\vec{p}_\perp'-\vec{p}_\perp$. To see how this can be important, let us label all hadron states by an index N that includes all intrinsic quantum numbers including mass M, angular momentum J, etc., as well as an index h for helicity and a momentum index. The equations (1) can then be regarded, at $p_z=\infty$ and with $\vec{k}$ and $\vec{k}'$ perpendicular to the z axis, as equations for matrices with rows and columns labelled by N and h alone, without any further reference to the state of motion of the particles. This situation presents a mathematical analogy to the situation in nonrelativistic quantum mechanics. Consider, for example, a (presumably fictitious) "molecule" composed of a fixed number of quarks, labelled by a running index n. A representation of the algebra (1) is obtained if we take[12]

$$F_i(\vec{k}_\perp)\sim U^{-1}[\textstyle\sum_n\frac{1}{2}\lambda_i{}^{(n)}\exp(i\vec{k}_\perp\cdot\vec{x}^{(n)})]U,$$

$$F_i^5(\vec{k}_\perp) \sim U^{-1}[\sum_n \tfrac{1}{2}\lambda_i^{(n)}\sigma_z^{(n)} \exp(i\vec{k}_\perp \cdot \vec{x}^{(n)})]U, \quad (2)$$

where U is an arbitrary unitary transformation, $\vec{x}$ is the position operator of a quark, and σ and λ are spin and unitary-spin matrices for the quarks. The labels N and h would here correspond to the labeling of a complete set of nonrelativistic wave functions for the quarks. The average momentum of initial and final states is not mentioned, only the momentum transfer.

In the actual relativistic problem with $p_z = \infty$, the operators $F_i(\vec{k}_\perp)$ and $F_i^5(\vec{k}_\perp)$ have complicated angular-momentum properties, which we describe below, and it is not clear that an elementary representation like (2) is appropriate, even with a very special form of the operator U. However, it is tempting to imagine that some relatively simple representation could describe approximately an idealized infinite set of baryon or meson bound states and resonances, while of course a very much larger representation actually describes exactly the huge space of all states, including any number of particles, with a given value of the total baryon number. In order to state the angular-momentum conditions, let us define not only the mass operator M but also the intrinsic parity operator $\mathcal{P}$ and an angular momentum[5] operator whose components are $\mathcal{J}_i$, with $\mathcal{J}_z = h$ and the matrix elements of $\mathcal{J}_x$ and $\mathcal{J}_y$ between helicity states equal to the usual matrix elements of J_x and J_y between eigenstates of J_z, while all matrix elements of the $\mathcal{J}_i$ between different values of N vanish. The equations we obtain among the operators M, $\mathcal{J}_i$, $F_i(\vec{k}_\perp)$, and $F_i^5(\vec{k}_\perp)$ should then be satisfied in the representation along with the commutation relations.[13] If such a small, approximate representation can be found and is successful, it would permit the rough calculation of the spectrum of baryon or meson levels, with their angular momenta and parities, and the matrix elements of weak and electromagnetic currents between them, including form factors; with the aid of approximate principles like partial conservation of axial-vector current, the pattern of strong coupling constants could also be predicted.

To deduce the angular-momentum properties of $F_i(k)$, we write

$$\langle N', h', | F_i(\vec{k}_\perp) | Nh \rangle = \langle N', h', p_x = k/2, p_y = 0, p_z = \infty | \mathfrak{F}_{i0}(0) | N, h, p_x = -k/2, p_y = 0, p_z = \infty \rangle, \quad (3)$$

where we have arbitarily taken $k_\perp$ in the x direction. We then transform[14] to a Breit frame and obtain[5]

$$\langle N'h' | F_i(\vec{k}_\perp) | Nh \rangle$$
$$= \mathfrak{N} \langle N', h', p_x = 0, p_y = 0, p_z = q/2 | \exp(-i\mathcal{J}_y \omega') Z_i(0) \exp(-i\mathcal{J}_y \omega) | N, h, p_x = 0, p_y = 0, p_z = -q/2 \rangle, \quad (4)$$

with

$$Z_i(0) = \mathfrak{F}_{i0}(0) - lq^{-1}\mathfrak{F}_{iz}(0) - kq^{-1}\mathfrak{F}_{ix}(0), \quad (5)$$

$$l = (M'^2 - M^2)(2M^2 + 2M'^2 + k^2)^{-1/2},$$
$$q = (k^2 + l^2)^{1/2}, \quad (6)$$
$$\omega = \arctan[2Mk(M'^2 - M^2 + k^2)^{-1}],$$
$$|\omega| \leq \pi/2 \ (>\pi/2) \text{ for } (M^{2\prime} - M^2 + k^2) \geq 0 \ (<0), \quad (7)$$
$$\omega'(M, M', k) = -\omega(M', M, -k)$$
$$\mathfrak{N} = 2[(M^2 + q^2)(M'^2 + q^2)]^{1/4} \times [(M^2 + q^2)^{1/2} + (M^{2\prime} + q^2)^{1/2}]^{-1}.$$

In the Breit frame, the properties of the matrix elements are well known and expressible[14] in terms of multipole form factors like the famous G_E and G_M. Inverting the rotations in Eq. (3), one finds that the helicity matrix $\langle N'h' | \exp(i\mathcal{J}_y \omega') F_i(k_\perp) \exp(i\mathcal{J}_y \omega) | Nh \rangle$ for fixed N and N' has $\Delta\mathcal{J}_z = 0, \pm 1$. The matrix $\langle N'h' | \times F_i(k_\perp) | Nh \rangle$ has the following properties: (i) The part odd in $k_\perp$ has $\Delta\mathcal{J}_z$ odd while the even part has $\Delta\mathcal{J}_z$ even. (ii) It is invariant under $\mathcal{P}\exp(i\pi\mathcal{J}_y)$ and $T\exp(i\pi\mathcal{J}_y)$. (iii) For $N = N'$, so that the Breit-frame momentum q is equal to k, a multipole expansion[12] shows that the coefficient of k^j contains only $|\Delta\mathcal{J}|$ with $1 \leq |\Delta\mathcal{J}| \leq j+1$ for odd j and only even $|\Delta\mathcal{J}| \leq j$ for even j. For a conserved current, of course, $F_i(0)$ has vanishing matrix elements unless $N' = N$, $h' = h$; also the coefficient of k^j

then has $|\Delta \mathcal{J}| \leq j$. In the case of $F_i^5(k_\perp)$, the current is never conserved; the only other differences are that the matrix element is odd rather than even under $\mathcal{P}\exp(i\pi\mathcal{J}_y)$ and that the coefficient of k^j has $1 \leq |\Delta\mathcal{J}| \leq j+1$ with $|\Delta\mathcal{J}|$ odd for odd j.

In order to see the content of the $\Delta\mathcal{J}_z = 0$, ± 1 condition, it is instructive to look first at the simplified case $\Delta\mathcal{J}_z = 0$. We write the $\Delta\mathcal{J}_z = 0$ condition as $[\mathcal{J}_z, \exp(i\mathcal{J}_y\omega')F_i(k_\perp)\exp(i\mathcal{J}_y\omega)] = 0$ which is equivalent to

$$K'F_i(\vec{k}_\perp) - F_i(\vec{k}_\perp)K = 0 \quad (\Delta\mathcal{J}_z = 0), \tag{8}$$

where the matrices K' and K with helicity rows and columns are given by

$$K' \equiv [(k^2+M^2-M'^2)\mathcal{J}_z + 2M'k\mathcal{J}_x] \times [k^4 + 2k^2(M^2+M'^2) + (M^2-M'^2)^2]^{-1/2},$$
$$K \equiv [(-k^2+M^2-M'^2)\mathcal{J}_z + 2Mk\mathcal{J}_x] \times [k^4 + 2k^2(M^2+M'^2) + (M^2-M'^2)^2]^{-1/2}. \tag{9}$$

The above equations are, of course, supposed to hold only when sandwiched between states with masses M' and M. Multiplying (8) by $[k^4 + 2k^2(M^2+M'^2) + (M^2-M'^2)^2]^{1/2}$, we obtain a rationalized expression which is easily converted to the operator equation

$$k^2\{\mathcal{J}_z, F_i(k)\} = [M^2, [\mathcal{J}_z, F_i(k_\perp)]] + k[2M\mathcal{J}_x, F_i(k_\perp)] \quad (\Delta\mathcal{J}_z = 0). \tag{10}$$

For the realistic case $\Delta\mathcal{J}_z = 0, \pm 1$, we begin by equating a triple commutator with $\mathcal{J}_z$ to a single commutator with $\mathcal{J}_z$ and arrive at

$$K'^3F_i(\vec{k}_\perp) - 3K'^2F_i(\vec{k}_\perp)K + 3K'F_i(\vec{k}_\perp)K^2 - F_i(\vec{k}_\perp)K^3 = K'F_i(\vec{k}_\perp) - F_i(\vec{k}_\perp)K \quad (\Delta\mathcal{J}_z = 0, \pm 1), \tag{11}$$

which can also be rationalized and written as an operator equation relating F_i, M, and the components $\mathcal{J}_i$.

We have found an interesting, though unphysical, illustrative example in which the angular properties can be imposed on a simple representation of the algebra. We consider a single quark with an infinite number of excited states and take Eq. (2) with $n = 1$ only, but we take all the states to be degenerate, with mass M. The transformation U is chosen so as to take $x \to (1/M)(Q_x + L_y)$ and $y \to (1/M)(Q_y - L_x)$, where

$$\vec{Q} = M\vec{x} - \vec{p}\cdot\vec{x}\vec{p}/2M - \vec{p}\vec{x}\cdot\vec{p}/2M, \quad \vec{L} = \vec{x}\times\vec{p}.$$

As one can verify, $\vec{L}$ and $\vec{Q}$ generate the algebra of the Lorentz group and it is that fact that permits Eq. (11) to be obeyed, but we have not so far found any way to generalize this method to a more physical case.

The notion we have discussed of trying to use a small representation of the infinite-order group generated by the charge densities is related to the idea[15-17] of using a small unitary representation of a finite-order but noncompact group. In both cases, we may try an idealized infinite level spectrum with either (a) "L-excitation," that gives high values of J_z by the case of high values of L_z, while the SU(3) representations are restricted to $\underline{1}$, $\underline{8}$, and $\underline{10}$ for the baryons and $\underline{1}$ and $\underline{8}$ for the mesons; or (b) "spin-isospin excitation," that gives high values of J_z by the use of more and more pairs of mathematical quarks, so that large SU(3) representations result, including exotic values of I and Y. At the moment, the experimental evidence for well-defined resonances with such values of I and Y is not impressive, and in the trivial example above we have considered case (a), but case (a) may turn out to be wrong either for experimental reasons or because our mathematical system has only solutions of a more complicated kind.

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[1]M. Gell-Mann, Phys. Rev. 125, 1067 (1962).

[2]M. Gell-Mann, Physics 1, 63 (1964).

[3]B. W. Lee, Phys. Rev. Letters 14, 271 (1965).

[4]R. Dashen and M. Gell-Mann, Phys. Letters 17, 275 (1965).

[5]R. Dashen and M. Gell-Mann, in Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1966 (W. H. Freeman & Company, San Francisco, California, 1966). It is important to keep in mind that $\mathcal{J}_i$ works only on the helicities of a state and does not change its momentum. For examples, we have $\exp(i\theta J_y)|h\vec{p}\rangle = \sum h' d_{hh'}^{(\theta)}|h'\vec{p}\rangle$.

[6]S. Fubini and G. Furlan, Physics 1, 229 (1964).

[7]S. Adler, Phys. Rev. Letters 14, 1051 (1965); Phys. Rev. 140, B736 (1965).

[8]W. Weisberger, Phys. Rev. Letters 14, 1047 (1965).

[9]S. Adler, Phys. Rev. 143, 1144 (1966).

[10]S. Fubini, to be published.

[11]We have obtained this result by explicitly constructing, in the Bargmann-Wigner formalism, the matrix

elements of the currents between particles of arbitrary spin and parity. Upon evaluating the matrix elements in the limit $p_z \to \infty$, one finds that none of the couplings depends on $p_\perp + p_\perp'$. This will not be true for any finite p_z. The origin of this fact is, of course, that due to relativity invariance, the dependence of the matrix elements on $p_\perp + p_\perp'$ is determined by the dependence on $p_0 + p_0'$ and $p_z + p_z'$. Since as $p_z \to \infty$ the dependence on the latter quantities, which become infinite, disappears, the limiting value of the matrix element cannot depend on $p_\perp + p_\perp'$.

[12]In Ref. 1, the similarity of the commutation relations to the nonrelativistic sum rules was pointed out, but not the fact that nonrelativistic quantum mechanics provides actual representations of the algebra.

[13]If this can in fact be done in the simple case (2), then we would represent $\mathcal{J}_i$ by the ith component of $\sum_n (\vec{x}^{(n)} \times \vec{p}^{(n)} + \vec{\sigma}^{(n)}/2)$ and M by an operator acting on the quark spins and coordinates. Apart from the interpretation, the irrelevance of the existence of real quarks, and the crucial operator U, this case represents the concrete-quark model of R. H. Dalitz [Proceedings of the Oxford International Conference on Elementary Particles, Oxford, England, 1965 (Rutherford High Energy Laboratory, Chilton, Berkshire, England, 1966)] and G. Morpurgo [Physics 2, 95 (1965)].

[14]L. Durand, P. DeCelles, and R. Marr, Phys. Rev. 126, 1882 (1962).

[15]Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Letters 17, 148 (1965).

[16]A. O. Barut, in Proceedings of the Seminar on High-Energy Physics and Elementary Particles (International Atomic Energy Agency, Vienna, Austria, 1965).

[17]C. Fronsdal, in Proceedings of the Seminar on High-Energy Physics and Elementary Particles (International Atomic Energy Agency, Vienna, Austria, 1965).

ERRATUM

CONTRIBUTION TO THE HYPERFINE FIELD FROM CATION-CATION INTERACTIONS. Nai Li Huang, R. Orbach, and E. Šimánek [Phys. Rev. Letters 17, 134 (1966)].

In our calculation of the "super" hyperfine interaction between Mn^{2+} ions in $KMnF_3$, we considered only the $^2p_\sigma$ orbitals of the surrounding F^- ions. We have subsequently included the $2s$ F^- orbitals and find a substantial reduction of the $3d \to 4s$ transfer coefficient. This change leads to a reduction of our estimate of ΔH_{hyp} from 29.8 to 12.4 kG. Using the hyperfine coupling constant quoted in our Ref. 5 and the results of Minkiewicz and Nakamura [Phys. Rev. 143, 356 (1966)], we now find the reduction in sublattice magnetization of $KMnF_3$ caused by zero-point spin-wave oscillations to be 3.4%. The details of this calculation will be included in a paper soon to be submitted to the Physical Review.