correlations in going from Ca^{40} to Ca^{44}. This could decrease the transition rates to the 3^- and 5^- states in Ca^{42} and Ca^{44} relative to Ca^{40} in approximately the same way. At the present time there is evidence for an effect of this type from stripping reactions.[16] This implies that more complicated particle-hole configurations will have to be taken into account in future calculations.

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[1]G. E. Brown, Unified Theory of Nuclear Models (North-Holland Publishing Company, Amsterdam, 1965). This book contains a description of the particle-hole model and references to the original literature.

[2]Vincent Gillet and E. A. Sanderson, Nucl. Phys. 54, 472 (1964).

[3]R. W. Bauer, A. M. Bernstein, G. Heymann, E. P. Lippincott, and N. S. Wall, Phys. Letters 14, 129 (1965).

[4]J. Saudinos et al., Compt. Rend. 252, 260 (1961); W. S. Gray, J. J. Kraushaar, and R. A. Kenefick, Nucl. Phys. 67, 565 (1965).

[5]D. Blum, R. Barreau, and J. Bellicard, Phys. Letters 4, 109 (1963).

[6]R. H. Bassel, G. R. Satchler, R. M. Drisko, and E. Rost, Phys. Rev. 128, 2693 (1962).

[7]L. P. Leenhouts and P. M. Endt, Phys. Letters 5, 69 (1963); M. A. Grace and A. R. Poletti, Nucl. Phys. 78, 273 (1966).

[8]The DWBA analysis has been performed following Ref. 6 in which a collective model of the nuclear states has been assumed. The strength of a state is taken as β_l^2, where β_l is the fractional root-mean-square deformation of the ground state. The relative values of β_l^2 are approximately equal to the relative cross sections. The absolute value of β_3 for Ca^{40} (3.73 MeV) is 0.24, in good agreement with the value found in the Ca^{40} (e,e') experiment.[5]

[9]In Ti^{50} only two 3^- states and no 5^- states were found. This may not be significant, however, since the target used was only 70% isotopically pure, whereas the Ca targets used were $\geq$95% pure.

[10]R. J. Peterson, Phys. Rev. 140, B1479 (1965); thesis, University of Washington (1966) (unpublished). The Ca targets used in this experiment were thin (particularly Ca^{48}) and contained large C and O contaminations which precludes proper identification of weaker states.

[11]K. Matsuda, Nucl. Phys. 33, 536 (1962).

[12]H. W. Broek, Phys. Rev. 130, 1914 (1963).

[13]R. Lombard, P. Kossangi-Demag, and G. R. Bishop, Nucl. Phys. 59, 398 (1964).

[14]J. Bellicard, P. Barreau, and D. Blum, Nucl. Phys. 60, 319 (1964); J. R. Meriwether, I. Gabrielli, D. L. Hendrie, J. Mahoney, and B. G. Harvey, Phys. Rev. 146, 712 (1966).

[15]E. J. Martens and A. M. Bernstein, to be published.

[16]T. A. Belote, A. Sperduto, and W. W. Buechner, Phys. Rev. 139, B80 (1965); E. Kashy, A. Sperduto, H. A. Enge, and W. W. Buechner, Phys. Rev. 135, B865 (1964); W. E. Dorenbusch, T. A. Belote, and Ole Hanson, Phys. Rev. 146, 734 (1966); T. A. Belote, J. Rapaport, and W. E. Dorenbusch, to be published.

ANOMALOUS ISOTOPE SHIFT OF THE NUCLEAR CHARGE RADIUS*

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It is shown that the isospin dependence of the shell-model potential gives rise to the experimentally observed "anomalous" isotopic shift of nuclear charge radii. The charge radii calculated from a Woods-Saxon potential are in qualitative agreement with experimental determinations from electron scattering and from muonic x rays.

The purpose of the present note is to point out that the isospin-dependent term in the optical potential[1] naturally brings about anomalies in the isotopic shift of the nuclear charge distribution, of the type that have been observed in a series of recent experiments on muonic x rays[2,3] and electron scattering[4] as well as in earlier results on the optical isotope shift.[5] This anomaly consists in the experimental observation that within the isotopes of one element

the charge radius does not increase as rapidly as $A^{1/3}$ but at a rate which is considerably smaller. This seems to imply that the charge distribution does not follow this most simple expectation on a microscopic scale, even though throughout the periodic table it does follow the $A^{1/3}$ pattern. We wish to point out that such a behavior of the charge distribution is to be expected when computed from wave functions of single-particle states in a Woods-Saxon potential including an isospin-dependent term.

It is well known that proton binding energies vary among isotopes of the same element and, in particular, that the binding energy tends to increase with neutron excess. Lane noted that there should be an isospin-dependent term in the average potential that the nucleus presents to a proton (or neutron).[1] In the shell-model potential, this term explains the general trend in proton binding energies as the neutron excess increases in isotopes of the same element. The increase in binding energy will cause the wave function of the more tightly bound protons to penetrate less far into the negative-energy region of larger radii and will give rise to a smaller average radius. Also, in a diffuse potential, the radius will get smaller because of the slope in the sides of the potential-energy well.

In the investigation reported here, the calculations started with a proton potential which fits the scattering of protons from many nuclei and which contains a dependence on neutron excess.[6] For a Woods-Saxon shape, the parameters of this potential are $R = 1.25A^{1/3}$ F and $a = 0.65$ F. A recent experiment on the elastic scattering of 12-MeV protons by Ca isotopes[7] yields optical potentials whose parameters are consistent with the isospin dependence of the potential of Ref. 6. We, however, chose to determine the well depth required for calculating the charge distribution by requiring that the binding energy of a $1f_{7/2}$ proton be correctly given. These energies are, in fact, the differences between the ground-state energies in the pairs Sc^{41}-Ca^{40}, Sc^{45}-Ca^{44}, etc.[8] The potentials resulting from such a search were then used to bind the filled proton states. Mean-square radii were computed for each state and averaged with proper weights to give root-mean-square radii of the total proton distribution.

The results of such calculations are given in Table I together with similar results for the even Ca and Ni isotopes. The fractional difference in charge radii $[R_c(Ca^{44})-R_c(Ca^{40})]/R_c(Ca^{40})$ is calculated as -1.55%, to be compared with the $+3.23\%$ change in $A^{1/3}$ and the average experimental value[2] of $(+1.1 \pm 0.3)\%$. In Table II, the parameters for the charge distribution obtained from electron scattering on Ca isotopes[4] are compared with the present calculations. The changes in radius and diffuseness were obtained by fitting the Fermi shape to the calculated charge distribution. It is apparent that the increased binding causes the deviation from $R_c \propto A^{1/3}$ and the decrease in the diffuseness of the charge distribution.

Another source of experimental charge radii for Ca isotopes is the measured value of Coulomb energies, obtained from the positions of isobaric-analog states. These energies have been used to extract radii[9] of Ca isotopes by assuming a uniform spherical charge distribution; the results are summarized in Table III. It is clear that all the experimental data bear out the predictions from our model qualitatively: The charge radii increase less rapidly than $A^{1/3}$ and the diffuseness decreases with neutron

Table I. Charge radii calculated by matching known binding energies.

Nucleus	$\langle R_c^2\rangle^{1/2}$ (F)	Percent difference from lightest isotope	Equivalent $R_c A^{-1/3}$ for uniformly charged sphere (F)
Ca^{40}	3.319	...	1.253
Ca^{42}	3.287	-0.97	1.221
Ca^{44}	3.268	-1.55	1.195
Ca^{48}	3.293	-0.78	1.170
Ni^{58}	3.685	...	1.229
Ni^{60}	3.685	0.0	1.215
Ni^{62}	3.726	1.11	1.216
Ni^{64}	3.716	0.82	1.199

excess. Some deviations, such as occur in Ca^{42} and Ca^{44}, may well have their source in more specific nuclear-structure effects not included in our calculation. The difference between Ca^{48} and Ca^{40}, the two closed-shell nuclei for which perhaps our model should work best, is described remarkably well.

Hahn, Hofstadter, and Ravenhall[10] studied the elastic scattering of 183-MeV electrons from Ni^{58} and Ni^{60} and observed a difference which was consistent with an increase in the charge radius. But since they could not obtain absolute cross sections, they could not positively identify the source of the difference in the charge distributions. In the present calculation, in which the radius of the potential well increases as $A^{1/3}$, the effect of the increased binding in Ni^{60} exactly cancels the effect of the increase in size of the well, so that the rms radii are the same.

A similar calculation carried out for the Ca isotopes matched the binding energies of the last $d_{3/2}$ proton. These energies are obtained from the differences in ground-state energies of K^{39}-Ca^{40}, K^{43}-Ca^{44}, etc. Such a calculation indicates that the change in charge radius in going from Ca^{40} to Ca^{44} is -1.74%, which is not very different from the -1.55% calculated from the binding energies of the next proton. Also, the same degree of difference in charge radii is obtained when one varies the geometrical parameters of the Woods-Saxon potential in the vicinity of the established values.

In the above calculations, the isospin term of the potential was assumed to have the same Woods-Saxon shape as the rest of the potential (i.e., the scalar part). The increase in radius, which is kept proportional to $A^{1/3}$, is supposed to represent the increase in the scalar part of the potential. There is no a priori reason to believe that the two parts of the potential should have the same radial distribution, and some other radial form for the symmetry term should be investigated. Another convenient shape for the form factor is the derivative of the Woods-Saxon shape (the symmetry term peaked at the surface of the nucleus). Then, to first order, the effect of the added isospin term is just to increase the radius of the well. It has long been known that the elastic scattering is rather insensitive to an increase in the radius of the optical-model potential if it is compensated by a decrease in the potential depth. Because of the well-known VR^2 ambiguity in optical potentials, the results of Ref. 6 could just as well be interpreted in terms of a radius change as a function of $(N-Z)/A$ instead of an increase in well depth. The analysis of direct (p,n) reactions to analog states[11] does, in fact, indicate a preference that an isospin term in the potential should be concentrated near the surface. One may then expect that the mean neutron radius will be larger than the proton radius in heavy nuclei.[12]

To match the single-particle energies, we may therefore adjust the radius of the potential well of Ca^{44} with respect to the one for Ca^{40} by more than the usual $A^{1/3}$ dependence. In the calculation summarized in Table I, the depth

Table II. Parameters describing the charge distribution in Ca isotopes. The values from the present calculation are compared with the ones derived from electron scattering in Ref. 4. The parameters c and $r_{0.5}$ are radii, and t and Z are measures of the diffuseness, all defined in Ref. 4.

	Charge radius		Diffuseness	
	Obs.	Calc.	Obs.	Calc.
	C (F)		Z (F)	
Ca^{40}	3.60 ± 0.043	3.617	0.576 ± 0.029	0.506
	$\Delta C/C$ (%)		$\Delta Z/Z$ (%)	
Ca^{44}	2.18 ± 0.07	1.55	-1.66 ± 0.82	-10.0
Ca^{48}	4.1 ± 0.1[a]	3.62	-12.4 ± 1.0[a]	-13.0

[a]Because of the use of an additional parameter in the shape of the charge distribution, the values of $\Delta r_{0.5}/r_{0.5}$ and $\Delta t/t$ are given.

Table III. Charge radii from Coulomb energies in analog states.

Isotope	From Coulomb energies[a]	From present calculation	From $R_c = R_{c_0} A^{1/3}$
		R_c (F)	
Ca^{40}	4.189	4.285	...
		$[R_c(A) - R_c(40)]/R_c(40)$ (%)	
Ca^{42}	+2.5	-0.97	+1.64
Ca^{44}	-0.1	-1.55	+3.23
Ca^{48}	-1.4	-0.78	+6.27

[a]See Ref. 7. The uncertainties in these radii are $\pm 0.5\%$, from experimental uncertainties only.

of the Woods-Saxon potential for Ca^{40} was increased by 3.83 MeV to give the observed binding energy in Ca^{44}. The observed binding energy can also be matched by keeping the well depth fixed at the Ca^{40} value and increasing the radius parameter from $1.25A^{1/3}$ F for Ca^{40} to $1.3125A^{1/3}$ F for Ca^{44}. Instead of -1.74% for the change in the radius of the charge distribution for Ca^{44}, one now obtains $+3.24\%$. This result is interesting since it may constitute evidence that the form factor for the isospin term is not entirely peaked at the surface of the nucleus, a result which may well be consistent with Ref. 11.

It is difficult to estimate how good a quantitative agreement one should expect from such studies since many of the implicit assumptions in the calculations are not completely justified. The binding energy of the next (or last) proton is not a very good measure of the potential because of the rearrangement energy; and the Woods-Saxon shape is an approximation in any case. Furthermore, the potential is expected to be velocity dependent (or nonlocal). This last point may not be too serious since a study[13] of the spacing of single-particle states near the top of the Fermi sea seems to indicate an effective mass of approximately unity. This is in apparent contradiction with many calculations on infinite nuclear matter and the known energy dependence of the optical-model potential. We have also neglected the finite size of the proton charge in our calculations.

In view of the above reservations as to the justification of such calculations, it is interesting to compare the general trend of calculated results for the equivalent charge radii with their experimentally determined values from electron scattering and muonic x-ray data. The charge radii shown in Fig. 1 were determined by an empirical fit to the electron-scattering data by Elton[14] and from muonic x rays.[15] The theoretical curve was obtained from a Woods-Saxon potential with $R = 1.25A^{1/3}$ F and $a = 0.65$ F, with the well depth adjusted to give the last-proton binding energy, and with the spin-orbit potential 25 times the Thomas strength (using the pion mass). The trend as a function of atomic weight is clearly reproduced. Similar calculations, but much more restricted in scope, have been previously reported.[16]

For heavier nuclei it may be difficult, if not impossible, to detect the slight variations of the density that such calculations predict in

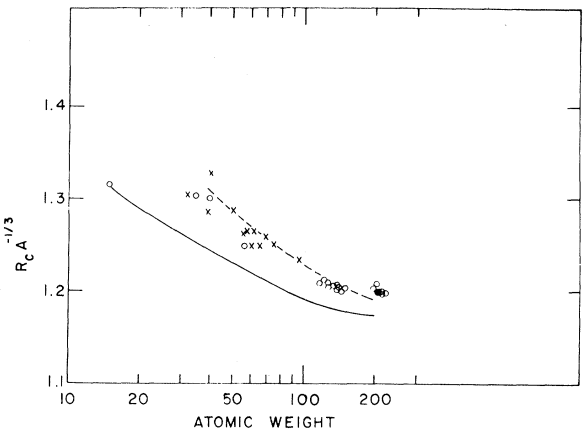

FIG. 1. Equivalent charge radius [rms radius times $(5/3)^{1/2}$] as a function of mass number. The crosses, dots, and square are the muonic x-ray data of the three papers mentioned in Ref. 15 in that order. The dashed line represents Elton's empirical curve for electron scattering. The solid line represents the trend in present calculations.

the interior of the nucleus. But it is quite possible that the large changes in the central density associated with the filling-in of the $2s_{1/2}$ proton shell between Si and S and the $3s_{1/2}$ shell between Au and Pb could be detected. Figure 2 shows the magnitude of the effect for closure of the $3s_{1/2}$ shell. The analysis of the electron-scattering data for gold[17] and mercury[18] shows, as calculated, that a central depression in the charge-density distribution is slightly favored by the data. It is not known whether or not the Pb data indicate the filling in of the central depression observed in Au, as obtained from this calculation.

To conclude, it seems that the charge den-

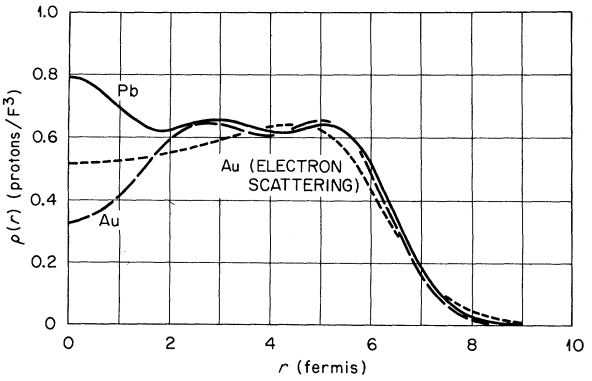

FIG. 2. Distribution of charge density as a function of radius near the filling of the $3s_{1/2}$ proton single-particle state.

sity calculated from a Woods-Saxon potential which is in agreement with the optical model and reproduces the last-particle binding energy agrees with the experimental determinations. In particular, it explains the experimental observation that charge radii do not increase as rapidly as $A^{1/3}$ within the isotopes of one element.

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[1]A. M. Lane, Phys. Rev. Letters 8, 161 (1962).

[2]H. L. Anderson, C. S. Johnson, E. P. Hincks, S. Raboy, and C. C. Trail, Phys. Letters 6, 261 (1963); J. A. Bjorkland, S. Raboy, C. C. Trail, R. D. Ehrlich, and R. J. Powers, Phys. Rev. 136, B341 (1964); R. C. Cohen, S. Devons, A. D. Kanaris, and C. Nissim-Sabat, Phys. Letters 11, 70 (1964).

[3]C. Chasman, R. A. Ristinen, R. C. Cohen, S. Devons, and C. Nissim-Sabat, Phys. Rev. Letters 14, 181 (1965); H. L. Anderson, R. J. McKee, C. K. Hargrove, and E. P. Hincks, Phys. Rev. Letters 16, 434 (1966); T. T. Bardin, R. C. Barrett, R. C. Cohen, S. Devons, D. Hitlin, E. Macagno, C. Nissim-Sabat, J. Rainwater, K. Runge, and C. S. Wu, Phys. Rev. Letters 16, 718 (1966).

[4]M. Croissiaux, R. Hofstadter, A. E. Walker, M. R. Yearian, D. G. Ravenhall, B. C. Clark, and R. Herman, Phys. Rev. 137, B865 (1965); R. Hofstadter, C. K. Nöldeke, K. J. van Oostrum, L. R. Suelzle, M. R. Yearian, B. C. Clark, R. Herman, and D. G. Ravenhall, Phys. Rev. Letters 15, 758 (1965); K. J. van Oostrum, R. Hofstadter, G. K. Nöldeke, M. R. Yearian, B. C. Clark, R. Herman, and D. G. Ravenhall, Phys. Rev. Letters 16, 528 (1966).

[5]For a summary of experimental results see P. Brix and H. Kopferman, Rev. Mod. Phys. 30, 517 (1958).

[6]F. G. Perey, Phys. Rev. 131, 745 (1963).

[7]A. Marinov, L. L. Lee, Jr., and J. P. Schiffer, Phys. Rev. 145, 852 (1966).

[8]It should be remarked, however, that many of the effects that can cause the binding energies of the last proton to change would not be reflected in the average well for all the bound protons. Such would be odd-even differences or other effects caused by configurations involving a few nucleons. Remaining within one set of isotopes and considering the binding of a single-particle state materially reduces this possible difficulty.

[9]K. W. Jones, J. P. Schiffer, L. L. Lee, Jr., A. Marinov, and J. L. Lerner, Phys. Rev. 145, B894 (1966).

[10]B. Hahn, R. Hofstadter, and D. G. Ravenhall, Phys. Rev. 105, 1353 (1957).

[11]G. R. Satchler, R. M. Drisko, and R. H. Bassel, Phys. Rev. 136, B637 (1964).

[12]We have carried out some calculations for the neutron distribution in the Ca isotopes in a manner analogous to that used for protons, except that the energy of the $2p_{3/2}$ neutron state was used to determine the well depth. The results indicate that the neutron radius of Ca^{48} is 11.9% greater than that of Ca^{40} and is 5.6% more than expected from an $A^{1/3}$ dependence. The net nucleon radius, however, changes by 6.6%, which is very close to an $A^{1/3}$ dependence. Very recently an analysis of elastic-scattering data revealed differences in the radii of the proton and neutron distributions similar to the ones reported here [G. W. Greenlees, G. J. Pyle, and Y. C. Tang, Phys. Rev. Letters 17, 33 (1966)].

[13]G. E. Brown, H. H. Bunn, and P. Gould, Nucl. Phys. 46, 598 (1963).

[14]L. R. B. Elton, Nuclear Sizes (Oxford University Press, London, England, 1961). This monograph contains an extensive review of the literature on this subject prior to 1960.

[15]D. Quittman, R. Engfer, U. Hegel, P. Brix, G. Backenstoss, K. Goebel, and B. Stadler, Nucl. Phys. 51, 609 (1964); H. L. Acker, G. Backenstoss, C. Daum, J. C. Sens, and S. A. DeWit, to be published; H. L. Anderson, R. J. McKee, C. K. Hargrove, and E. P. Hincks, Phys. Rev. Letters 16, 434 (1966).

[16]A. E. S. Green, K. Lee, and R. J. Berkeley, Phys. Rev. 104, 1625 (1956); A. A. Ross, Hans Mark, and R. D. Lawson, Phys. Rev. 102, 1613 (1956); R. R. Shaw, A. Swift, and L. R. B. Elton, Proc. Phys. Soc. (London) 86, 513 (1965); D. F. Jackson, Proc. Phys. Soc. (London) 76, 949 (1960).

[17]D. G. Ravenhall and D. R. Yennie, Phys. Rev. 96, 239 (1954); B. Hahn, D. G. Ravenhall, and R. Hofstadter, Phys. Rev. 101, 1131 (1956).

[18]G. E. Brown and L. R. B. Elton, Phil. Mag. 46, 164 (1955).