Table II. Critical temperatures and slopes of the critical field curves at $T=T_c$ for the isotopes of zinc.

	T_c (°K)	$(dH_c/dT)_{T=T_c}$ (Oe/°K)
Zn^{64}	0.856 ±0.0015	104.5 ±1.6
Zn^{66}	0.848 ±0.0015	104.1 ±1.7
Zn^{68}	0.841 ±0.0015	103.4 ±1.7

$dT)_{T=T_c}$ obtained from the work reported here. The errors shown for these quantities in Table II and the errors given for z and ξ in Table I reflect the extreme spread of these values for different runs. They also reflect differences introduced by assuming that the temperature at which each transition takes place is that at which the signal had decreased by 90% of its maximum change instead of by 50%.

*Work supported by the U. S. Atomic Energy Commission and the Wisconsin Alumni Research Foundation.

[1]J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 108, 1175 (1957).

[2]J. C. Swihart, IBM J. Res. Develop. 6, 14 (1962).

[3]P. Morel and P. W. Anderson, Phys. Rev. 125, 1263 (1962).

[4]J. W. Garland, Jr., Phys. Rev. Letters 11, 114 (1963).

[5]J. W. Garland, Jr., thesis, University of Chicago (unpublished).

[6]T. H. Geballe and B. T. Matthias, IBM J. Res. Develop. 6, 256 (1962).

[7]M. W. Garrett, J. Appl. Phys. 22, 1091 (1951).

DOMAIN VELOCITY, STABILITY, AND IMPEDANCE IN THE GUNN EFFECT

B. W. Knight* and G. A. Peterson
United Aircraft Research Laboratories, East Hartford, Connecticut
(Received 27 April 1966)

This Letter presents a framework for analyzing space-charge waves in two-valley semiconductors. We have (1) determined different classes of propagating waves, (2) developed velocity criteria for accumulation and depletion layers, (3) analyzed the stability of either domains or layers, and (4) derived an expression for the ac impedance across a propagating domain.

Undimensionalized, the relevant laws are as follows[1-3]:

Displacement current,

$$\mathcal{E}_t(x,t)=-j(x,t)+f(t); \tag{1}$$

internal current law,

$$j(x,t)=n(x,t)v(\mathcal{E})-D(\mathcal{E})n_x(x,t); \tag{2}$$

Poisson's equation,

$$\mathcal{E}_x(x,t)=n(x,t)-1, \tag{3}$$

where f is the external current, and the drift velocity $v(\mathcal{E})$ is given in Fig. 1(a). With new coordinates moving at velocity c ($\xi=x-ct$), the above give the partial differential equation

$$\mathcal{E}_t+(v-c)\mathcal{E}_\xi-D\mathcal{E}_{\xi\xi}=f-v. \tag{4}$$

Without diffusion ($D=0$), Eq. (4) can be solved exactly[3] and shows the development of domains[4] bounded by shock fronts where $|\mathcal{E}_{\xi\xi}|\to\infty$.

For constant external current, Eq. (4) admits steady propagating solutions ($\mathcal{E}_t=0$). This case can be reduced to quadratures.[2] Letting $\mathcal{E}_\xi=\mathcal{E}_\xi(\mathcal{E})$ (for which $\mathcal{E}_{\xi\xi}=\mathcal{E}_\xi d\mathcal{E}_\xi/d\mathcal{E}$) yields the relationship

$$\Phi(\mathcal{E},\mathcal{E}_\xi)\equiv\mathcal{E}_\xi-\ln(\mathcal{E}_\xi+1)-\int_{\mathcal{E}_\alpha}^{\mathcal{E}}d\mathcal{E}'\frac{v(\mathcal{E}')-c}{D(\mathcal{E}')}=K+(c-f)\int_{\mathcal{E}_\alpha}^{\mathcal{E}}\frac{d\mathcal{E}'}{D(\mathcal{E}')[\mathcal{E}_\xi(\mathcal{E}')+1]}, \tag{5}$$

where K is a constant of integration. If $c=f$, then $\Phi=K$ becomes a first integral. Once $\mathcal{E}_\xi(\mathcal{E})$ is known, the spatial waveform follows from quadrature of $d\mathcal{E}/d\xi=\mathcal{E}_\xi(\mathcal{E})$.

Various classes of solutions are shown by the curves in the $(\mathcal{E},\mathcal{E}_\xi)$ phase-plane diagram Fig. 1(b): (1) The open arc curves Γ_1 and Γ_2 both attach to the singular points $\mathcal{E}_\alpha$ and $\mathcal{E}_\gamma$ and represent, respectively, accumulation and depletion layers; (2) the curve Γ_3 is for $c=f$ and $K=0$. Initially $\mathcal{E}_\xi=0$ while $\mathcal{E}=\mathcal{E}_\alpha$; then $\mathcal{E}_\xi$ goes through positive values until $\mathcal{E}$ achieves a maximum, and then through negative values as $\mathcal{E}$ returns to $\mathcal{E}_\alpha$. This is a high-field propagating domain. A family of low-field domains attached to $\mathcal{E}_\gamma$ exists for external currents in-

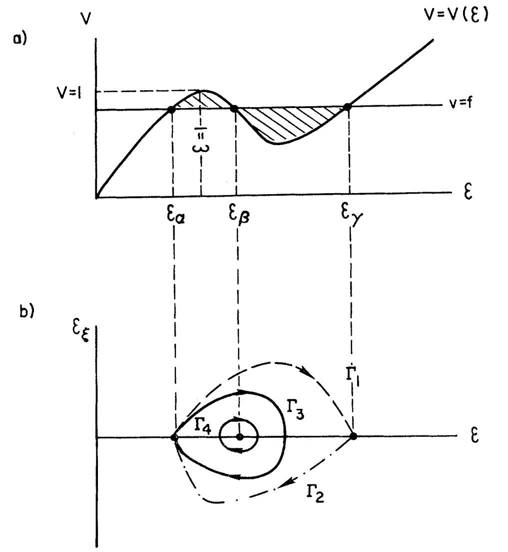

FIG. 1. (a) $v(\mathcal{E})$ with anomalous region, (b) orbits in the $(\mathcal{E}_\xi,\mathcal{E})$ phase plane.

sufficient to support high-field domains. (3) The curve Γ_4 is for another value of the invariant; it is representative of a family of solutions which cannot attach to a singular point and corresponds to a periodic oscillation. This situation corresponds to multiple propagating domains, as observed experimentally by Kino and Owen.[5]

A closed orbit is impossible for $c \neq f$,[2] since the contrary would imply

$$\oint d\mathcal{E}\{D(\mathcal{E})[\mathcal{E}_\xi(\mathcal{E})+1]\}^{-1/2}=0, \tag{6}$$

from the single valuedness of $\Phi(\mathcal{E},\mathcal{E}_\xi)$. But the integrand of (6) is manifestly smaller on the outward arc than on the return. However, the open arc or layer solutions are permissible for $c \neq f$. [Closed orbit solutions with more complicated internal current laws then (2) may likewise depart from the criterion $c=f$.]

Where $\mathcal{E}_\xi=-1$, from Eq. (4) $\mathcal{E}_{\xi\xi}$ has a single signature; hence the system point can cross $\mathcal{E}_\xi=-1$ only once; thus for any path between singular points,

$$\mathcal{E}_\xi>-1, \tag{7}$$

which will be used below.

To simplify the presentation subsequent developments will be for field-independent diffusion. There is a natural generalization for $D(\mathcal{E})$.

A cluster of rigorous results follow from a general perturbation on Eq. (4)

$$[\mathcal{E}=\mathcal{E}_0(\xi)+\mathcal{E}_1(\xi,t),\ f=f_0+f_1(t),\ c=c_0+c_1]$$

$$\text{yielding } \mathcal{E}_{1t}+L\mathcal{E}_1=f_1+c_1\mathcal{E}_{0\xi}, \tag{8}$$

where

$$L=-\left\{D\frac{\partial^2}{\partial\xi^2}-(v-c_0)\frac{\partial}{\partial\xi}-(\mathcal{E}_{0\xi}+1)v_{\mathcal{E}}\right\}. \tag{9}$$

The translational invariance of Eq. (4) gives at once

$$L\mathcal{E}_{0\xi}=0. \tag{10}$$

Thus $\mathcal{E}_{0\xi}$ is an eigenfunction of L, with eigenvalue zero.

For a criterion on the velocities of "layer" solutions, let $f_1=\text{const}$, $\mathcal{E}_1=\mathcal{E}_1(\xi)$ in Eq. (8), giving

$$L\mathcal{E}_1=f_1+c_1\mathcal{E}_{0\xi}. \tag{11}$$

A solution to (11) exists only if there is no admixture, on the right-hand side, of the eigenfunction of L corresponding to eigenvalue zero. This condition determines c_1. Under the similarity transformation

$$T(\xi)=\exp\left[-\tfrac{1}{2}\int^{\xi}d\xi'\frac{(v-c)}{D}\right], \tag{12}$$

the operator

$$\bar{L}=TLT^{-1}, \tag{13}$$

becomes Hermitian with orthogonal eigenfunctions. The admixture can be evaluated directly as a Fourier coefficient, and setting it equal to zero yields

$$\frac{dc}{df}=-\frac{\int_{-\infty}^{\infty}d\xi\,T^2\mathcal{E}_{0\xi}}{\int_{-\infty}^{\infty}d\xi\,T^2(\mathcal{E}_{0\xi})^2}. \tag{14}$$

Thus $dc/df<0$ for $\mathcal{E}_{0\xi}>0$; and $dc/df>1$ if $\mathcal{E}_{0\xi}<0$ follows from (14) and (7). A value of f can be chosen such that $c=f$ for both layer solutions. This occurs if $\Phi(\mathcal{E}_\gamma,0)=\Phi(\mathcal{E}_\alpha,0)=0$, which is the condition that the two hatched areas in Fig. 1(a) are equal.[2] Thus for higher external currents the dc/df conditions above show that the accumulation layer moves more slowly, and the depletion layer moves faster than the drift velocity.

The stability of the various wave forms against perturbations which do not alter the external current will be determined by choosing $\mathcal{E}_1(\xi,t)$

$=\exp(\lambda t)\mathcal{E}_1(\xi)$, $f_1=c_1=0$ in Eq. (8), giving

$$\lambda\mathcal{E}_1=-L\mathcal{E}_1. \qquad (15)$$

Stability exists if and only if all the eigenvalues of L are non-negative. The eigenfunctions of L and $\tilde{L}$ [Eq. (13)] have the same zeroes and eigenvalues. $\tilde{L}$ is Hermitean; its "ground state" is the one eigenfunction with no zeroes. For "layer" modes, $\mathcal{E}_{0\xi}$ has no zeroes; hence these modes will be stable. For the "domain" mode on the other hand, $\mathcal{E}_{0\xi}$ has a zero, implying another eigenfunction with negative eigenvalue, and hence, instability. Similarly for the periodic mode, a whole "energy band" of eigenvalues less than zero exist, giving instability.

Finally, by Kirchoff's laws, the ac impedance, Z, of an operating diode equals that of the external circuit. In Eq. (8) this gives

$$f_1=\frac{1}{Z}\int_a^b d\xi'\mathcal{E}_1(\xi'), \qquad (16)$$

and $c_1=0$. Here $\mathcal{E}$ is understood to include a dipole domain well removed from both ends of a diode of length $(b-a)$. The time-dependence eigenvalue equation becomes

$$(\lambda+L)\mathcal{E}_1=\frac{1}{Z}\int_a^b d\xi'\mathcal{E}_1(\xi'). \qquad (17)$$

Such an equation may be solved by specifying the eigenvalue (say $\lambda=i\omega$) and solving for the "coupling constant" Z.[6] If the eigenfunctions $\mathcal{E}_{1m}$ of L are so normalized that $\int T^2\mathcal{E}_{1m}{}^2 d\xi=1$, the solution is

$$Z(\omega)=\sum_m(\lambda_m+i\omega)^{-1}\left(\int d\xi\,\mathcal{E}_{1m}\right)\left(\int d\xi\,T^2\mathcal{E}_{1m}\right). \qquad (18)$$

For a diode containing a propagating domain, λ_0 is negative, as observed above, the $m=1$ term vanishes with the first parenthesis of Eq. (18), and higher terms become characteristic of the passive regions of the diode. Thus, approximately (neglecting distributed capacitance of the passive diode), Eq. (18) may be written in the form

$$z(\omega)=\frac{(1/C)}{(1/R_-C)+i\omega}+R_+, \qquad (19)$$

where in our units

$$R_+=(b-a)/v_{\mathcal{E}}(\mathcal{E}_\alpha), \qquad (20)$$

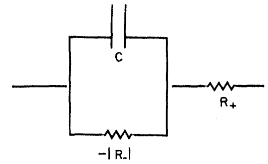

FIG. 2. Equivalent circuit for sample containing a domain.

which is positive,

$$C=\left(\int d\xi\,\mathcal{E}_{10}\right)^{-1}\left(\int d\xi\,T^2\mathcal{E}_{10}\right)^{-1}, \qquad (21)$$

which is positive, and

$$R_-=1/C\lambda_0, \qquad (22)$$

which is negative. This leads to the equivalent circuit shown in Fig. 2.

An inspection of Eq. (17) reveals two periodic modifications in the domain: an oscillation in its thickness and an oscillation in displacement from its equilibrium path. The former is responsible for the impedance given in Eq. (19).

If the external impedance $Z(\omega)$ is specified, Eq. (18) or (19) becomes a stability equation to be solved for complex $i\omega$. A positive real part implies instability. For example, in the special case of fixed external voltage ($Z=0$), Eq. (19) yields

$$i\omega=-[(1/R_+C)+(1/R_-C)]. \qquad (23)$$

One interesting implication is that a domain becomes unstable if the external load is too large.

We would like to thank Professor A. McWhorter for drawing our attention to the work of Butcher.[2]

*Permanent address: The Rockefeller University, New York, New York, and Cornell University Graduate School of Medical Sciences, New York, New York.

[1]B. K. Ridley and T. B. Watkins, Proc. Phys. Soc. (London) 78, 293 (1961); B. K. Ridley, ibid. 82, 954 (1963); C. Hilsum, Proc. IRE 50, 185 (1962); H. Kroemer, Proc. IEEE 52, 1736 (1964); D. E. McCumber and A. G. Chynoweth, IEEE Trans. Electron. Devices ED13, 4 (1966).

[2]P. N. Butcher, Phys. Letters 19, 546 (1965).

[3]B. W. Knight and G. A. Peterson, Phys. Rev. (to be published).

[4]B. K. Ridley, Proc. Phys. Soc. (London) 82, 954 (1963).

[5]G. Kino and J. Owen, private communication.

[6]B. W. Knight and G. A. Peterson, Phys. Rev. 132, 1085 (1963).