inhomogeneous term and the purely geometric kernel K. We believe this is better done for specific cases where the spin and isospin symmetries of the interactions can be directly utilized to simplify the geometric structure at an earlier stage, and do not attempt to give a general formula here.

We wish to emphasize that these are now well-defined integral equations in two continuous variables with a maximum of $3(L+1)\times\min(2J+1, 2L+1)$ components, and that the dynamical singularities of the two-body interactions have been explicitly separated, insofar as is physically allowable, from the purely geometrical coupling between the three interacting subsystems.

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WICK ROTATION IN THE BETHE-SALPETER EQUATION*

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Interest in the two-particle Bethe-Salpeter (B.S.) equation[1] has been revived recently.[2] One reason for this is that the separable approximation to a generalization of the Faddeev equations[3] gives rise to such an equation, albeit for resonance-particle scattering.[4] But even in the two-particle scattering region this equation includes inelastic effects which cannot be taken account of by N/D equations; nor can they be included if the Bethe-Salpeter kernel is replaced by the Blankenbecler-Sugar kernel.[5]

There are two main difficulties to be faced in attempting a numerical solution to the B.S. equation: a large number of variables and numerous singularities of the kernel. The number of variables can be reduced to a minimum of two by a particle-wave expansion; therefore, it is necessary to remove the singularities before reasonably accurate computations can be performed on present computers. In this Letter we wish to give a systematic and practical method to remove completely all the singularities in the B.S. kernel. This is an extension of Wick rotations[6] into the elastic, simply inelastic, etc., regions which takes account not only of displaced poles but also of displaced cuts. We start with the full B.S. equation for scattering of two spinless particles of mass m via the exchange of a particle of mass μ:

$$M(q,q'';p)=g^2[(q-q'')^2-\mu^2]^{-1}+\frac{ig^2}{(2\pi)^4}\int d^4q'[(q-q')^2-\mu^2]^{-1}[(p-q')^2-m^2]^{-1}[(p+q')^2-m^2]^{-1}M(q',q'';p). \qquad (1)$$

We take $p=(\frac{1}{2}\sqrt{s},\vec{0})$, where $\sqrt{s}$ is the invariant total energy. We work in momentum space as distinct from coordinate space, since renormalization and three-body equations are handled more naturally in p space.

In order to perform a Wick rotation we study the singularities of the integrand in (1) in the variable q_0' $[q'=(q_0',\vec{q}\,')$, etc.]. We evidently have six poles arising from the three propagators at $q_0'=\pm p_0\pm(\vec{q}\,'^2+m^2)^{1/2}$, $q_0\pm[(\vec{q}-\vec{q}\,')^2+\mu^2]^{1/2}$ (where the Feynman $i\epsilon$ is used). Let us call the first set of propagator poles the direct poles, the second set exchange poles. There are also singularities in q_0' arising from the function $M(q'q'';p)$ itself. These singularities are composed of two branch lines starting at $-p_0+[\vec{q}\,'+(m+\mu)^2]^{1/2}$ and going to $+\infty$ just below the real axis, and from $p_0-[\vec{q}\,'^2+(m+\mu)^2]^{1/2}$ to $-\infty$ above the real axis. They arise from pinches between the first- and second-type poles mentioned above. These branch lines contain higher branch points at $q_0'=\pm\omega_r$, $\omega_r=p_0-[\vec{q}\,'^2+(m+r\mu)^2]^{1/2}$. We will refer to these cuts as inelastic cuts.

Most of these singularities will be removed after the contour of integration in q_0' is rotated counterclockwise to the imaginary axis and $M(q_0)$ is analytically continued to the imaginary axis in q_0. This rotation may pick up residues from displaced poles and discontinuities from cuts protruding into the first and third quadrants. It will not remove true singularities (pinches) if their position is independent of q_0. We must consider the location of these singularities before rotation, in particular, their dependence on q_0, p_0, and $|\vec{q}'|$. They come from pinches between two direct poles, direct and exchange pole, etc. There are evidently five types which we denote by dd, de, di, ei, and ii.[7] The particular pinches which can occur in a given range of $p_0=\frac{1}{2}s^{1/2}$ are given in Table I. We note that the de and ie pinches generate the higher branch points on the inelastic cuts; also that dd, di, and ii do not depend on q_0 but only on p_0 and $|\vec{q}'|$. Thus they cannot be removed by the above rotation while the de and ie pinches are.

If we are only interested in the bound-state region there are no such displaced contributions,[6] neither poles nor cuts. In the elastic region [(2) of Table I] and for all "higher" p_0 there is a displaced pole d. There is also a pinch dd singularity which is independent of q_0 and so not removed by rotation; it will have to be removed by subtraction, as discussed elsewhere.[8,9] In region (3) there is the additional di pinch which is not removed by rotation; it is the logarithmic singularity generated by the exchanged particle. It will have to be removed by a further subtraction. In region (4) the new phenomenon of a displaced cut occurs; the contribution from this cut cannot necessarily be neglected.

The contribution coming from this displaced cut which persists from region (4) onward can be taken in account correctly by using an integral representation (EAR)[10] for M, embodying the q_0 analyticity. The EAR representation is

$$M(q_0\vec{q};p_0q'')=M_0+\int_{-\infty}^{\alpha_-}dt\frac{m_-(t,\vec{q};p_0,q'')}{t-q_0}+\int_{\alpha_+}^{\infty}dt\frac{m_+(t,\vec{q};p_0,q'')}{t-q_0}, \qquad (2)$$

where $\alpha_+(\vec{q},p_0)=-p_0+[\vec{q}^2+(m+\mu)^2]^{1/2}$, $\alpha_-=-\alpha_+$, and M_0 is the first Born term. We insert the EAR in Eq. (1) and perform the q_0' integration. The discontinuity of the resulting integrals in q_0 is taken and then its partial-wave projection is performed. The resulting equations for $m_{\pm l}$, with $\vec{m}=(m_{+l},m_{-l})$ are

$$\vec{m}=\vec{m}^0+\mathsf{K}\cdot\vec{m}, \qquad (3)$$

where

$$\mathsf{K}(q_0,q;tq')=\frac{g^2}{8\pi^2}\frac{q'}{q}\begin{pmatrix}K_{++}K_{+-}\\K_{-+}K_{--}\end{pmatrix} \qquad (4)$$

Table I. The table shows, in columns 1 and 2, position and q_0, q_0' dependence of the singularities which can occur in the different regions of p_0. Column 3 shows the effect of the Wick rotation on the singularities. Column 4 lists poles and cuts protruding into the first quadrant, n, s, t count the number of exchange particles present.

Range of p_0	Pinches	Effect of Wick rotation on singularities	Displaced poles and cuts
(1) $p_0<m$ (bound state)	None	Removes all	None
(2) $m<p_0<m+\frac{1}{2}\mu$ (elastic)	dd: $p_0=(q'^2+m^2)^{1/2}$	Not removed	Displaced d pole
	de, ie: $p_0=q_0\pm\{[q'^2+(m+n\mu)^2]^{1/2}+[(\vec{q}-\vec{q}\,)^2+\mu^2]^{1/2}\}$	Removed	
(3) $m+\frac{1}{2}\mu<p_0<m+\mu$	dd: $p_0=(\vec{q}'^2+m^2)^{1/2}$	Not removed	
	di: $2p_0=[q'^2+m^2]^{1/2}+[q'^2+(m+n\mu)^2]^{1/2}$	Not removed	
	ie, de: as above	Removed	
(4) $m+\mu<p_0$	dd, di, de, ie: as above	As above	Displaced d pole and i cut
	ii: $2p_0=[q'^2+(m+s\mu)^2]^{1/2}+[q'^2+(m+t\mu)^2]^{1/2}$	Not removed	

and

$$K_{+-}=\frac{P_l(z_1)\theta(1-z_1^2)}{8\omega p_0(p_0-\omega)(p_0-\omega+t)}+\frac{P_l(z_2)\theta(1-z_2^2)}{8\omega p_0(p_0+\omega)(p_0+\omega-t)},$$

$$K_{++}=K_{+-}-\frac{P_l(z_3)\theta(1-z_3^2)}{[(t-p_0)^2-\omega^2][(t+p_0)^2-\omega^2]}. \quad (5)$$

Here we have set $\omega^2=\vec{q}'^2+m^2$, and

$$2qq'z_i=-a_i^2+\vec{q}^2+\vec{q}'^2+\mu^2,$$

$$a_1=q_0+p_0-\omega,$$

$$a_2=q_0-p_0-\omega,$$

$$a_3=q_0-t, \quad (6)$$

with equivalent expressions for K_{--} and K_{-+}. Also

$$m_{+l}^0=\frac{g^4}{8\pi^2}\int\frac{q'}{q}dq'dt\left\{\frac{\theta(1-z_2^2)P_l(z_2)}{8p_0\omega(p_0+\omega)[(p_0+\omega-q_0'')^2-\omega'^2]}+\frac{\theta(1-z_1^2)P_l(z_1)}{8p_0\omega(p_0-\omega)[(p_0-\omega-q_0'')^2-\omega'^2]}+\frac{\theta(1-z_6^2)P_l(z_6)}{2\omega'[(p_0+\omega'+q_0'')^2-\omega^2][(p_0-\omega'-q_0'')^2-\omega^2]}\right\},$$

where $a_6=q_0-\omega'-q_0''$, $\omega'^2=(\vec{q}'-\vec{q}'')^2+\mu^2$.

The poles of the kernels in (5) are the only singularities left in the equations for the $\vec{m}$ functions. We observe that they do not depend on the external variables $q_0, \vec{q}$ and so can be removed by the subtraction method developed earlier by one of us.[8] We see that the EAR has explicitly performed the Wick rotation; it has removed all the singularities depending on $q_0, \vec{q}$. In energy range (1) no singularity remains; there is a singularity in $\vec{q}'$ in range (2) when $p_0=\omega$, $\omega^2=\vec{q}'^2+m^2$. It arises from the dd pinch; we remove it here by a standard subtraction[11] in the spectral representation. The further singularities arising in K are in regions (3) and (4) at $t=\pm p_0\pm\omega$. These can be removed by further subtractions, and the resulting coupled integral equations have kernels with no singularities in them (are locally bounded functions with suitable decrease at infinity).[12] We will present more details, applications, and numerical calculations in different regions elsewhere.

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