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DISPERSION OF ELECTRON PLASMA WAVES*

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The effect of plasma temperature on the dispersion of electron-plasma waves is an old problem. Shortly after the plasma oscillations in an infinite, cold plasma were described by Langmuir and Tonks,[1] the effect of random thermal motion on their dispersion relation was calculated by Thomson and Thomson.[2] They assumed a resonance distribution for the electron velocities and obtained a result differing from that of later work by a numerical coefficient. Vlasov[3] studied the problem using a Maxwellian velocity distribution and obtained, for the long wavelength limit, the dispersion relation

$$\omega^2 \simeq \omega_p^{\,2}[1+3(k\lambda_D)^2], \qquad (1)$$

where ω is the wave frequency, $\omega_p = (4\pi ne^2/m_e)^{1/2}$ is the plasma frequency, k is the wave number and λ_D is the Debye length. However, his work has been severely criticized[4] because of the rather cavalier manner with which he dealt with a divergent integral in the derivation. A generally accepted derivation of the dispersion relation has been given by Landau.[4] Landau also found that the waves were damped, even in the absence of collisions, and this part of the theory has been verified in detail experimentally.[5] The physical mechanism responsible for the dispersion and damping of the waves has been elucidated by Bohm and Gross.[6] All these calculations were made for an infinite, uniform collisionless plasma.

In this Letter we report an experiment designed to measure the effect of plasma temperature on the dispersion of electron plasma waves. When the theory is modified to include the effects of finite size of the system and spatial variation in plasma density, it accurately predicts the observed dispersion.

The geometry to be considered is a long column of plasma bounded in the radial direction by a good conductor. The plasma is immersed in a uniform finite magnetic field parallel to the axis of the plasma column. The plasma density is a function of radius, but its temperature is not. We specify the properties of the plasma by the dielectric tensor relating the displacement to the electric field ($\vec{D}=\epsilon\cdot\vec{E}$).[7] In the limit of waves with phase velocity small compared with the velocity of light, $\nabla\times\vec{E}$ may be neglected and the electric field calculated from a scalar potential, Ψ. When the plasma is regarded as a dielectric, there are no free charges and

$$\nabla\cdot\vec{D}=\nabla\cdot(\epsilon\cdot\vec{E})=-\nabla\cdot(\epsilon\cdot\nabla\Psi)=0. \qquad (2)$$

The frequency is sufficiently high that the motion of ions may be ignored. The electron gyroradius is assumed small compared with $\Psi(d\Psi/dr)^{-1}$ and $n(dn/dr)^{-1}$, where n is the electron density, so the dielectric tensor is a local quantity. Since n is a function of radial position, ϵ is also. In addition ϵ is a function of the wave frequency and the magnetic field.

A suitable solution for Ψ is

$$\Psi = \varphi(r)e^{i(kz+m\theta)}, \qquad (3)$$

where k is the complex wave number describing the wave. Substituting (3) into (2), one obtains

$$\frac{\partial^2\varphi}{\partial r^2}+\left[\frac{1}{r\epsilon_{rr}}\frac{\partial}{\partial r}(r\epsilon_{rr})\right]\frac{\partial\varphi}{\partial r}+\left[\frac{im}{r}\frac{\partial\epsilon_{r\theta}}{\partial r}\right]\varphi-\left[\left(\frac{m}{r}\right)^2+k^2\frac{\epsilon_{zz}}{\epsilon_{rr}}\right]\varphi=0, \qquad (4)$$

where ϵ_{zz} and ϵ_{rr} are components of the dielectric tensor.

The eigenfunction, φ, must satisfy the boundary conditions that φ equals zero at the conducting wall, and φ equals some given finite normalization factor at the origin. In general, φ is complex. We have solved Eq. (4) numerically, subject to the given boundary conditions, for the experimentally observed radial electron-density distribution. The complex eigenvalues, $k=k_r+ik_i$, computed for a series of real frequencies ω, give the dispersion and Landau damping of the waves.

The machine that produces the plasma has been described in detail previously.[5] The steady-state plasma is produced in a duoplasmatron-type hydrogen-arc source and drifts from it into a long uniform magnetic field of a few hundred gauss. For the data reported here, the resulting cylindrical plasma column has a length of 230 cm and a central density of 2.2×10^8 electrons/cm^3, and is immersed in a magnetic field of 183 G. The background pressure is 1.1×10^{-5} Torr (mostly H_2). The plasma temperature, kT (where k is Boltzmann's constant), may be adjusted by adjusting the source pressure, and it is 6.5 and 9.6 eV for the two cases discussed. Hence, the Debye length is about 1 mm. The electron mean free path for electron-ion collisions is of the order of 1000 m and for electron-neutral collisions is about 40 m. The plasma is surrounded by a stainless-steel tube 5.2 cm in radius, which acts as a waveguide beyond cutoff to reduce electromagnetic coupling between probes. The radial density profile of the plasma used for the present experiment, as measured by a Langmuir probe, is given in Fig. 1.

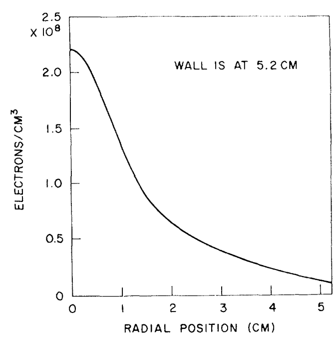

FIG. 1. Electron density as a function of radial position.

The plasma column is terminated at the end opposite to the source by a negatively charged plate. The electric field of this plate reflects the electrons. An electron velocity analyzer is mounted behind a 1-mm diam hole in the end plate. When the end plate is made somewhat less negative, electrons in line with the hole and with enough energy to reach the end plate pass into the analyzer. By means of a series of grids, the analyzer rejects any ions which enter it and measures current of electrons with energy greater than a given adjustable value. We obtain a current

$$I(E_0)=e\int_{E_0}^{\infty}F(E)dE, \qquad (5)$$

where $F(E)$ is the electron-energy distribution function and e is the electron charge. The quantity $\ln I(E_0)$ plotted electronically against E_0 is a straight line, proving that the parallel distribution function is decreasing exponentially with energy, i.e., it is Maxwellian in the velocity range corresponding to three to five times the mean thermal velocity. The parallel electron temperature of the plasma may be obtained directly from the slope of the curve. The uncertainty in the measured temperature is about 10% and mostly due to uncertainties in analyzer calibration (which was done with an electron beam).

Two radial probes, each consisting of a 0.2-mm diam tungsten wire, are placed in the plasma. One probe is connected by coaxial cable to a chopped-signal generator. The other probe is connected to a receiver which includes a sharp, high-frequency filter, a string of broad-band amplifiers, an rf detector, a video amplifier, and a coherent detector operated at the transmitter chopping frequency. Provision is made to add a reference signal from the transmitter to the receiver rf signal, i.e., we may use the system as an interferometer. The transmit-

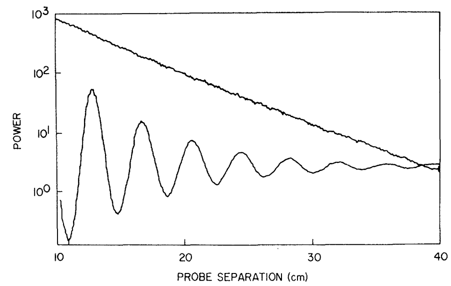

FIG. 2. Raw data. Upper curve is the logarithm of received power. Lower curve is interferometer output. Abcissa is probe separation.

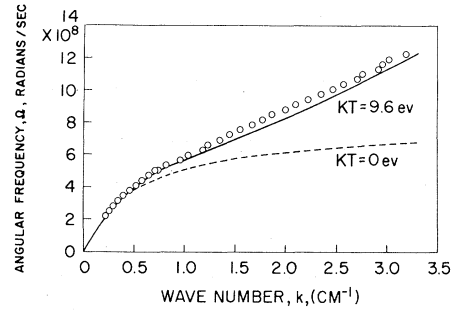

FIG. 3. Dispersion curve. The dashed line is computed assuming the plasma temperature is zero. The solid line is computed using the measured plasma temperature. The circles are experimental.

ter is set at a series of fixed frequencies, and at each, the receiving probe is moved longitudinally. The position of the receiving probe, which is transduced, is applied to the x axis of an x-y recorder, and the interferometer output or the logarithm of the received power is applied to the y axis.

Typical raw data are shown in Fig. 2. The slope of the power curve is the rate of power damping of the wave. The signal decreases smoothly as the probe is retracted radially with a half-maximum diameter about equal to that of the density profile. The distance between peaks on the interferometer curve is the wavelength. (There is a small correction because the waves are damped.) The wavelength can be determined to 3% over most of the range of the experiment. From the measured wavelengths and the transmitter frequencies we plot the dispersion relation of the waves, Fig. 3. The difference in dispersion curves for transmission in opposite directions is small.

For comparison with theory, we compute the dispersion relation for the wave from Eq. (4) using the measured radial density distribution, the plasma temperature measured by the velocity analyzer, and the experimental value of the magnetic field. We believe our Langmuir probe data are reliable for relative measurements of density, but not a sufficiently accurate measure of the absolute density for our purpose. Hence, we choose an absolute density which normalizes the theory to the experimental dispersion data at low frequencies (high phase velocities). In this part of the curve the temperature correction is negligible. This density is consistent with that obtained from probe measurements. The result is the solid curve of Fig. 3. For our geometry, for $k \gtrsim 1$ the slope of the curve is almost proportional to the mean thermal velocity of the plasma and is insensitive to the radial-density-distribution function. In this region we have almost an "infinite plasma" result. In the range $k \lesssim 1$ the dispersion is dominated by the finite size of the system. The small systematic difference between experiment and theory can be completely eliminated by using in the theory a mean thermal velocity about 5% larger than that obtained from the velocity-analyzer measurement, and this is within the accuracy of the analyzer calibration. In addition, the analyzer measures the distribution function at three to five times the mean thermal velocity. This is the region of interest for the damping measurement, but the dispersion depends strongly on the distribution function at lower velocities where the "temperature" may be slightly different. Both the Maxwellian character of the velocity distribution at high velocities and a consideration of the dynamics of electrons in the machine led us to believe that such an effect cannot be very large, but it could easily explain 5% differences in the effective mean thermal velocity.

At first it might appear that the curve should be parabolic in k as predicted by Eq. (1). However, the $(k\lambda_D)^2$ term in this equation is only the first term in an asymptotic series. For our case, the first four or five terms of this series are important, and the series does not even converge very well. To obtain satisfactory agreement, it is necessary to use the plasma-dispersion function, not an asymptotic representation. The difference between computed dispersion curves for the actual magnetic field of 183 G and for a magnetic field of infinity is negligible in this case. When the dispersion is computed, assuming that the plasma temperature is zero, the dashed curve of Fig. 3 is obtained. Except for a negligible correction due to the fact that the magnetic field is not infinite, this curve just scales vertically with the square root of absolute density. Since the slope of the experimental data is very different from that of the zero-temperature theory, it is impossible to obtain satisfactory agreement with the zero-temperature theory no matter what normalization of the absolute density of the plasma is chosen. The theoretical curve has been computed for another case in which the plasma temperature was 6.5 eV, and the measured dispersion curve again agrees well with the theory.

The Landau damping of these waves has been reported in detail in a previous publication.[5] We there showed that they exhibit heavy exponential damping under conditions where collisional damping is negligible. That the damping is caused by electrons traveling at the phase velocity of the wave, and the magnitude of the damping, its dependence of phase velocity, and its dependence on plasma temperature are accurately predicted by the theory of Landau.

In Ref. 5, we calculated the damping by expanding the solution of the eigenvalue equation around real k to obtain the small imaginary part of k. That procedure has the advantage that all dependence of the damping on the finite geometry of the system is eliminated in favor of the group velocity, a measured quantity. In addition, the plasma frequency does not appear in the formula either.[9] Using the present method of numerical integration, we again obtain the imaginary part of k and may compare it to the observed damping. The result is shown in Fig. 4. The theoretical result is essentially identical to that obtained by the expansion theory for $k_i/k_r < 0.05$. At $k_i/k_r = 0.08$ there is a 25% difference between the two methods of calculation due to higher order terms which have been left out of the expansion calculation.

There are a double infinity of solutions of Eq. (4) corresponding to various radial and angular eigenmodes. However, all higher modes at a given frequency are very heavily damped compared with the lowest mode, i.e., that with angular symmetry and the simplest radial dependence. Hence, when we apply a given frequency to the transmitting antenna, only the lowest mode is observable a short distance away, and only its properties are measured.

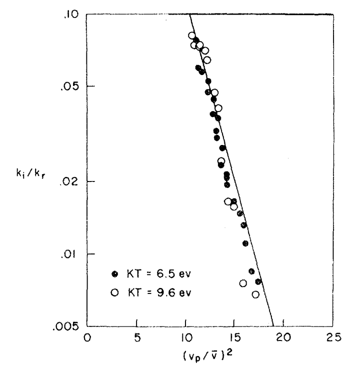

FIG. 4. k_i/k_r vs $(v_p/\bar{v})^2$. Solid curve is the theory by Landau.

In summary, the dispersion and damping of the electron plasma waves in our experiment may be computed by straightforward application of the theory. If the effect of plasma temperature is omitted from the theory, the theoretical result disagrees with the measurement.

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