of the asymptotic renormalized propagator, $\tilde{\Delta}_{C,R}'^{-1}(k) \sim Zk^2$, $k \to \infty$. An equivalent statement is obtained from the vacuum expectation value of the equal-time commutation relation,

$$\langle[\phi(\vec{x},x_0), \partial_0\phi(\vec{y},x_0)]\rangle = i\delta(\vec{x}-\vec{y})Z^{-1}, \qquad (2)$$

and the substitution of (1) into (2), with the use of the equal-time commutation relation appropriate to the elementary φ, yields

$$Z^{-1} = 4\langle\varphi^2\rangle / \langle\varphi^2|p\rangle^2. \qquad (3)$$

For any interaction of the φ field, the ratio on the right-hand side of (3) may be expected to diverge, providing the desired result, $Z=0$. Essentially just this divergent ratio was used by Nishijima[4] to demonstrate the validity of (1) in a particular model. Quite generally, $\langle\varphi^2\rangle$ may be written as $\mu_0{}^2N^2$, where μ_0 denotes the bare mass of the φ field, and N is a (linearly) divergent number increasing faster than any (typically, logarithmic) divergence of $\langle\varphi^2|p\rangle$; only for $\mu_0=0$ is this argument invalid.

Generalizations of these remarks to the case of a boson field composed of elementary fermion fields are fairly straightforward, although the specific form of the results will depend upon the interactions adopted for the fundamental fermion fields.

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NEW THEORETICAL VALUES FOR THE LAMB SHIFT*

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The general methods of application of renormalization theory and the degree of accuracy obtained in explicit calculations in the field of quantum electrodynamics have been in a state of continual refinement. This has been especially true for the theoretical calculation of the radiative corrections to the bound-state energy levels of light nuclei. The object of a large part of this effort has been to determine as accurate a comparison as possible of the experimental and theoretical values of the Lamb shift.[1]

This paper is concerned with the exact evaluation of the fourth-order radiative corrections to the energy levels of hydrogenic atoms. More precisely, the object here is to compute exactly those parts of the calculation that were previously estimated[2] and to recalculate and provide checks by various methods on all parts of the calculation. In terms of numbers the most recent experimental value of the Lamb shift[3] is 1058.05 ± 0.10 Mc/sec and the most recent theoretical value[4] is 1057.64 ± 0.21 Mc/sec. Of the total theoretical uncertainty, ± 0.10 Mc/sec is due to the estimated fourth-order radiative corrections. An exact calculation of these effects, therefore, would make possible a closer comparison of the experimental and theoretical values of the Lamb shift.

It has been rigorously shown[5] that the correct answer to this bound-state problem to order $\alpha^2(Z\alpha)^4$ can be obtained by first calculating the fourth-order radiative corrections to the elastic scattering of an electron in a fixed pointlike Coulomb potential in first Born approximation. The Feynman diagrams of interest here are shown in Fig. 1. Let M^γ denote the matrix elements of the corresponding diagrams where it is understood that

$$M^2 = M^{2'} + M^{2''} = 2M^{2''}, \quad M^4 = M^{4'} + M^{4''} = 2M^{4'}.$$

Let m_γ be that part of M^γ which is dimensionless and is defined by the relation

$$M^\gamma = -8\pi^2 e\alpha^2 \int d^4P_1 d^4P_2 \bar{\psi}(P_1)$$
$$\times \gamma_\mu A_\mu{}^e(P_1-P_2)[(P_1-P_2)^2/\kappa^2]\psi(P_2)m_\gamma,$$

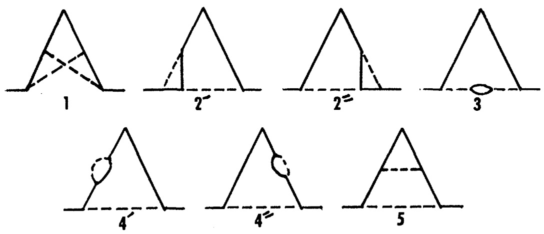

FIG. 1. Feynman diagrams that were calculated for the fourth-order radiative corrections to the scattering of an electron in a Coulomb field.

where P_2 and P_1 are the initial and final electron four-momenta, κ is the electron mass, $\bar{\psi}(P_1)$ and $\psi(P_2)$ are the free particle-electron wave functions, and $A_\mu^e(P_1-P_2)$ is the external four-component field potential. The value of e is taken as positive in rationalized Gaussian units and $\hbar$ and c are set equal to unity. Then from the scattering calculation one can infer a modified potential from which one can compute the following energy-level displacement formula[2]:

$$\frac{(Z\alpha)^4}{\pi^3}\mathrm{Ry}\left[\frac{4}{\pi^2}m\right]\delta_{l,0}, \qquad (1)$$

where m is the sum of the previously defined m_γ over all the diagrams.

One can write down the matrix elements for the scattering diagrams using the Feynman rules. The four reducible diagrams can be simplified by insertion of the renormalized second-order functions. The effect of this replacement is to leave only one integral over the photon momentum variable and either one or two integrals, depending on which function is involved, over the auxiliary variables that were introduced in the renormalization process of the second-order functions. In diagrams 3-5, no less than the following two methods were used to do the integrals over the photon momentum and auxiliary variables. The first way immediately combined all denominators and therefore introduced integrals over two or three additional auxiliary variables. The integral over the photon-momentum variable was done first, together with any further renormalization that had to be done in the fourth-order matrix element, and then the remaining integrals over the auxiliary variables were performed. The second way immediately integrated either one or two of the auxiliary variables that were contained in the renormalized second-order functions, thereby reducing the number of integrations to be performed at the outset. Additional auxiliary variables were introduced in order to combine the remaining denominators, and then the integration over the photon-momentum variable was performed. The integration over the remaining auxiliary variables completed the process. Additional methods consisted of different ways in which integrals could be separated or combined or proved equal by considerations of symmetry.

In each method of integration described in the previous paragraph there were two ways employed to perform the Dirac algebra required to extract the Lamb-shift terms. The integration over the photon-momentum variable involves at one point a linear transformation from the old to the new variable corresponding to a shift of the origin in momentum space. The first method expresses the shift in terms of q and s defined by

$$q=P_1-P_2, \quad s=P_1+P_2.$$

This transformation is then substituted into the numerator, the gamma sums completed, and the Lamb-shift terms calculated. The second method immediately completes all the gamma sums and expresses the shift in terms of P_1 and P_2. This transformation is substituted into the numerator, and then the Lamb-shift terms are extracted. Diagram 1 is irreducible and, therefore, integrations over two photon-momentum variables must be done. The denominators are combined and integrations over five auxiliary variables are introduced. A shift in each momentum variable is necessary to diagonalize the denominator, and the Dirac algebra of the numerator is calculated in the two ways described above. Eventually a large set of four-dimensional integrals over the auxiliary variables is obtained which are classified according to a system which makes the calculation and checking of their values a very organized and automatic task. Diagram 2, although reducible, can be handled in a way such that the final integrals over the auxiliary variables fit exactly into the classification scheme of those of diagram 1. Again the Dirac algebra is evaluated in the same way as previously discussed for diagram 1.

If λ is the photon mass then the final numerical values are

$$m_1=-\frac{13}{9}\ln\frac{\lambda}{\kappa}-\frac{\pi^2}{16}-\frac{7}{3}L-\frac{7}{6}\xi(3)+\frac{1613}{864},$$

Table I. Theoretical and experimental values of the Lamb shift in Mc/sec for H, D, and He^+.

	H	D	He^+
Theoretical	1057.499 ± 0.11	1058.763 ± 0.17	14038.17 ± 4.4
Experimental	1057.77 ± 0.10[a]	1059.00 ± 0.10[a]	14040.2 ± 4.5[b]
Experimental	1058.05 ± 0.10[c]	1059.34 ± 0.10[d]	

[a]Sol Triebwasser, Edward S. Dayhoff, and Willis E. Lamb, Jr., Phys. Rev. 89, 98 (1953).
[b]Edgar Lipworth and Robert Novick, Phys. Rev. 108, 1434 (1957).
[c]R. T. Robiscoe and B. L. Cosens, Phys. Rev. Letters 17, 69 (1966).
[d]R. T. Robiscoe and B. L. Cosens, Bull. Am. Phys. Soc. 11, 62 (1966).

$$m_2 = -\frac{2}{3}\ln^2\frac{\lambda}{\kappa} - \frac{1}{18}\ln\frac{\lambda}{\kappa} + \frac{2869}{4320}\pi^2 - \frac{7}{120}L - \frac{1691}{120}\xi(3) + \frac{132\,709}{8640},$$

$$m_3 = -\frac{77}{432}\pi^2 + \frac{1099}{648},$$

$$m_4 = \frac{2}{3}\ln^2\frac{\lambda}{\kappa} + \frac{1}{18}\ln\frac{\lambda}{\kappa} - \frac{17}{36}\pi^2 + \frac{1109}{864},$$

$$m_5 = \frac{13}{9}\ln\frac{\lambda}{\kappa} + \frac{91}{216}\pi^2 - \frac{319}{432},$$

$$m = \frac{1609}{4320}\pi^2 - \frac{287}{120}L - \frac{1831}{120}\xi(3) + \frac{504\,607}{25\,920} = 0.215\,296,$$

where

$$L = \frac{\pi^2}{2}\ln 2 - \frac{5}{4}\xi(3),$$

and $\xi(3)$ is the Riemann zeta function. One notes that the sum of the five diagrams is infrared convergent. Using Eq. (1) the additional energy caused by this perturbation to the $2s$ level of hydrogen and singly ionized helium is 0.102 and 1.63 Mc/sec, respectively. The previously calculated estimates of these corrections gave 0.24 ± 0.10 and 3.9 ± 1.6 Mc/sec, respectively. The new exact numbers change the recent theoretical values given by Erickson and Yennie[4] for H, D, and He^+. Table I lists these new theoretical values together with two rows of experimental values which separate the old values of Triebwasser, Dayhoff, and Lamb from new measurements of Robiscoe and Cosens.

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