functions regular within the extent of the solid:

$$\vec{f}_q^{\,\mathrm{I}}(\vec{r}) = \mathrm{grad}\, W_q(\xi_2, \xi_3) y_{1q}(\xi_1),$$

and

$$\vec{f}_q^{\,\mathrm{II}}(\vec{r}) = \mathrm{grad}\,[W_q(\xi_2, \xi_3) y_{2q}(\xi_1)].$$

Substitution of these in (1) yields a matrix whose eigenvalues are

$$K + 4\pi/3.$$

For a spherical shell of outer and inner radii R^{I}, R^{II} two radial functions, r^l and $r^{-(l+1)}$, enter with the same spherical harmonic Y_l^m. The resulting eigenvalues are now[8]

$$K - \frac{2\pi}{3} = \pm \frac{2\pi}{l+\frac{1}{2}} \left[\tfrac{1}{4} + l(l+1) \frac{R^{\mathrm{II}}}{R^{\mathrm{I}}(2l+1)} \right]^{1/2}$$

The range of K is now twice what it was before: between $-4\pi/3$ and $8\pi/3$.

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[1]H. Fröhlich, Theory of Dielectrics (Clarendon Press, Oxford, England, 1958), pp. 153-5.

[2]R. Fuchs and K. L. Kliewer, Phys. Rev. 140, A2076 (1965.

[3]M. Born and K. Huang, Dynamical Theory of Crystal Lattices, (Clarendon Press, Oxford, England, 1956), Secs. 24.4, 23.3. Our treatment bears a direct, though somewhat lengthy, generalization to many-atomic and polarizable lattices for which Eq. (1) still gives the spatial dependence of the phonons.

[4]T. H. K. Barron, Phys. Rev. 123, 1995 (1961).

[5]The three possibilities appear to give all physically interesting solutions to (1); however, neither the completeness of the solutions nor the bulk modes are treated in this Letter.

[6]P. M. Morse and H. Feshbach, Methods of Theoretical Physics, (McGraw-Hill Book Company, Inc., New York, 1953), p. 831; see also Chap. 5 for discussion on separable coordinates and for notation.

[7]The lowest order nontrivial solution, with $l = 1$, is the one found in Ref. 1.

[8]Regarding the slab as the limiting case of thin spherical shell, this includes the result of Ref. 2.

SOLUBLE EXTENSION OF THE ISING MODEL

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In this note we wish to relate a somewhat trivial, but surprising, soluble extension of the multidimensional Ising model. Our extension was motivated by recent experiments on a real material, dysprosium aluminum garnet (DyAlG), which closely resembles an ideal three-dimensional Ising model,[1] except for what appeared to be one unfortunate detail: The electronic (Ising) spins are connected not just to their electronic neighbors, but also to their own dysprosium nuclei by a fairly large hyperfine interaction. The interacting nuclear isotopes are randomly distributed with a natural abundance of 18.9% for Dy^{161} (hyperfine coupling constant $A = 0.073°K$) and 25% for Dy^{163} ($A = 0.104°K$).[2] This physical system thus corresponds to a model Ising antiferromagnet disturbed by a magnetic field (the hyperfine-coupled nuclear spins) random in magnitude and position. The magnitude of the disturbance is not negligible, considering the low critical temperature of DyAlG ($T_N = 2.5°K$), and is indeed comparable in magnitude with any one of the Ising bonds!

Given this substantial random perturbation of the Ising spins, it is reasonable to expect a drastic effect on thermodynamic properties, especially critical-point phenomena. For example, an estimate on the basis of molecular field theory shows each spin in a different specific environment, and hence a broadening of the transition region. The smallest effect on the thermodynamic properties which one might reasonably expect is a broadening of the lamb-

da point, or perhaps a change in the order of the phase transition.

The experiments[1] show no such thing! On the contrary, the usual type of logarithmic peak is found to within about 3×10^{-3} °K of the Néel point. We studied this problem theoretically, and discovered that this was no accident. We shall show that the thermal properties of the perturbed system can be related exactly to the properties of an ideal Ising model without requiring any knowledge of the latter. This is fortunate because, apart from various numerical estimates,[3] no solution of the three-dimensional Ising model in closed form is known at the present time. However, even without such a solution, we easily prove that the singularity at T_N and other critical-point phenomena are essentially the same in the perturbed system as in the ideal one, and that the two will differ only by smoothly varying nonsingular functions of temperature.

If we denote the nuclear spins by I_i and electron (Ising) spins by σ_i, spin-spin interactions by J_{ij}, and the hyperfine bonds by A_i, the total Hamiltonian $\mathcal{H}$ is

$$\mathcal{H}=\mathcal{H}_0+\tfrac{1}{2}\textstyle\sum_i A_i\sigma_i^z I_i^z, \tag{1}$$

where $\mathcal{H}_0$ is the ideal Ising Hamiltonian,

$$\mathcal{H}_0=\sum_{(ij)} J_{ij}\sigma_i^z\sigma_j^z-\mu H\sum_i \sigma_i^z, \tag{2}$$

and we neglect the extremely small direct interactions between the nuclear spins themselves and with the external field. The Ising spins have only two values, $\sigma_i^z=\pm1$, but the nuclear spins may have various magnitudes I_i (including $I=0$), depending on which isotopic species is at the ith site. For the case of Dy, $I=\frac{5}{2}$ for the two isotopes Dy^{161} and Dy^{163} and $I=0$ for all other isotopes.

We now note that the partition function of the combined system, $Z=\mathrm{Tr}\{\exp(-\beta\mathcal{H})\}$, may be evaluated in two steps, decomposing the grand trace into a partial trace over the nuclear spins only, Tr_I, followed by a trace over electron spins Tr_σ.[4] Using the fact that only even powers in $(I_i^z)^n$ contribute to Tr_I and that $(\sigma_i^z)^{2n}\equiv1$, we find

$$\begin{aligned} Z&=\mathrm{Tr}_\sigma\{\exp(-\beta\mathcal{H}_0)\}\,\mathrm{Tr}_I\{\exp(-\beta\textstyle\sum_i\tfrac{1}{2}A_iI_i^z)\}\\ &=Z_0Z_I, \end{aligned} \tag{3}$$

where Z_0 is the partition function of the unperturbed Ising system and Z_I a nuclear partition function independent of the electron spins identically equal to the partition function of the same nuclei in fixed external fields A_i. Z_I is a smoothly varying, continuous function of temperature and any singularities in the electron-spin system will thus be unaffected by the presence of the nuclear spins, no matter how large their coupling to the electron spins may be! In particular, the nuclei will leave T_N unchanged and they will simply add a smoothly varying term to the Ising specific heat $C_0(T)$:

$$C(T)=C_0(T)+\sum_i \tfrac{1}{2}A_iI_i\frac{d}{dT}B_i(\beta A_i/2), \tag{4}$$

where $B_i(x)$ is the Brillouin function appropriate to a spin I_i. This contribution is smooth and analytic and the nature of the singularity is therefore unaffected in all respects! It is doubtful whether the present authors would have sought, and found, this exact result had the experiments not indicated it first. Our results do not preclude the possibility that part of the broadening of the transition observed in other non-Ising-like materials[5] may in fact be due to random nuclear fields. In this connection it would be interesting to show whether or not the small width of the specific-heat peak observed in DyAlG ($\sim3\times10^{-3}$ °K) is in fact due to the finite non-Ising-like terms ($A\times\sigma\times I\times$etc.) in the hyperfine interaction.

It is now simple to extend the above method of splitting the traces to predict the behavior of the nuclear-spin system. For the average nuclear magnetization we find

$$\begin{aligned} \langle I_i^z\rangle&=Z^{-1}\,\mathrm{Tr}_\sigma\{\exp(-\beta\mathcal{H}_0)\\ &\qquad\times\mathrm{Tr}_I[I_i^z\exp(-\beta\textstyle\sum_n\tfrac{1}{2}A_n\sigma_n^zI_n^z]\}\\ &=I_iM_0(T)B_i(\beta A_i/2), \end{aligned} \tag{5}$$

where $M_0(T)$ is the electron sublattice magnetization, which vanishes above T_N. Below T_N we may also expect a correlation between different nuclei via the electronic spins, and we find for the long-range order

$$\langle I_n^zI_m^z\rangle=a_{mn}I_mI_nB_m(\beta A_m/2)B_n(\beta A_n/2), \tag{6}$$

where a_{mn} is the electron-spin order param-

eter in the absence of the nuclear spins,

$$a_{mn} = Z_0^{-1} \mathrm{Tr}_\sigma\{\sigma_m^z \sigma_n^z \exp(-\beta\mathcal{H}_0)\}. \quad (7)$$

With the hyperfine interactions appropriate to Dy in DyAlG, this nuclear antiferromagnetic order should become appreciable below about 0.1°K.

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[1]B. E. Keen, D. Landau, and W. P. Wolf, to be published. For a summary of earlier work see W. P. Wolf, in Proceedings of the International Conference on Magnetism, Nottingham, England, 1964 (The Institute of Physics and the Physical Society, University of Reading, Berkshire, England, 1965), p. 555.

[2]Values of A calculated from the observed electronic g value and the results of A. H. Cooke and J. G. Park, Proc. Phys. Soc. (London) 69, 282 (1956), assuming the ratio A/g to be the same for Dy^{3+} in the acetate and the garnet.

[3]See, for example, M. E. Fisher and M. Sykes, Physica 28, 939 (1962).

[4]A similar factorization has previously been used by M. E. Fisher, Phys. Rev. 113, 969 (1959), in connection with decoration extensions of Ising models.

[5]See, for example, J. Skalyo, Jr., and S. A. Friedberg, Phys. Rev. Letters 13, 133 (1964); P. Heller and G. B. Benedek, Phys. Rev. Letters 8, 428 (1962); T. Yamamoto, in Proceedings of the International Conference on Phenomena in the Neighborhood of Critical Points, Washington, D. C. (to be published).

MAGNETO-OSCILLATORY CONDUCTANCE IN SILICON SURFACES

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We have observed Shubnikov-de Haas[1] oscillations in a two-dimensional electron gas in the (100) surfaces of p-type silicon inverted by an electric field perpendicular to the surface.

The experiments were made on the field-effect structures shown in the inserts in Fig. 1. An electric field was applied between the 100-Ω-cm p-type silicon substrate and the aluminum gate electrode, so that degenerate electrons were induced in the silicon surface. The conductance between the diffused n^+ source and drain electrodes was measured. The distance between these coaxial contacts was 10 μ and the circumference of the gap between them was 500 μ. We have shown by Hall measurements[2] on similar surfaces that surface trapping was insignificant so that above the threshold for conduction the surface carrier density could be calculated from $n = \kappa(V_g - V_0)/4\pi\delta e$ in cgs units, where κ is the silicon-dioxide dielectric constant, V_g and V_0 are the voltage applied to the gate electrode and the threshold gate voltage, respectively, e is the electronic charge, and δ is the thickness of the oxide. Measurements were made on samples for which the oxide thicknesses were 5330 Å and 1150 Å, and the results were consistent.

The conductance and transconductance were measured as functions of magnetic field perpendicular to the surface up to 93 kOe. The conductance decreased to one-half of its zero-field value at 35 kOe at 1.4°K, at a gate voltage of 15 V for the thin oxide sample. The over-all variation agreed well with the usual expression $[1+(\mu^2H^2/C^2)]^{-1}$ for magnetoconductance,[3] yielding a mobility of 3000 cm²/V sec. This equaled the mobility calculated from the conductance at zero magnetic field. It is

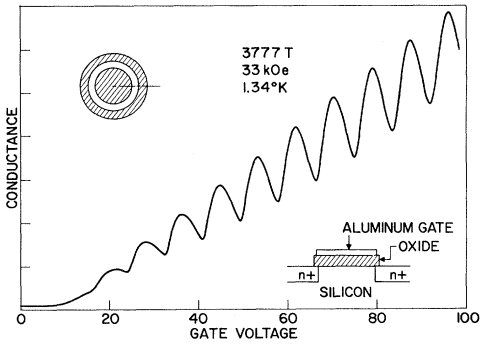

FIG. 1. The conductance as a function of gate voltage or surface field at 33 kOe. In the upper left-hand corner, a projection on the surface of the source-drain electrode configuration is shown. In the lower right-hand corner, a section through the structure is shown.