in which quarks have third-integral electric charges. The results obtained in this paper can be extended to many of the models with several triplets, possibly with integral charges. Let the members of any triplet in such a model be denoted by $p_i$, $n_i$, $\lambda_i$, where $p$, $n$ and $\lambda$ denote the SU(3) quantum numbers and the index $i$ labels the particular triplet. Then if the quark structure of mesons and baryons is such that Eq. (1) is modified only by putting indices on all the quark labels and adding summations in some places, the results presented here should be valid provided that appropriate assumptions are made regarding SU(3) symmetry.

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# RATIO OF THE WAVE-FUNCTION RENORMALIZATION CONSTANTS

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In this note we apply the method of Fubini and Furlan[1] to the equal-time canonical commutation relations of the renormalized fields for, say, pions and kaons. This method was recently used by Adler[2] and Weisberger[3] to obtain their celebrated sum rule for the axial-vector coupling-constant renormalization. Our main result here is $Z_3(K)/Z_3(\pi)\cong m_K^4/m_\pi^4$. Namely, the ratio of the wave-function renormalization constants of the kaon and the pion is equal to the fourth power of the ratio of the physical masses. Our arguments are by necessity heuristic; however, we shall not fail in showing at every stage explicitly what our assumptions are.

We start by writing down the canonical commutation relations for the renormalized pion and kaon fields, $\pi_\pm(x,t)$ and $K_\pm(x,t)$,

$$[\dot{\pi}_+(\vec{x},t),\pi_-(\vec{y},t)]=[-i/Z_3(\pi)]\delta(\vec{x}-\vec{y}),$$

$$[\dot{K}_+(\vec{x},t),K_-(\vec{y},t)]=[-i/Z_3(K)]\delta(\vec{x}-\vec{y}). \qquad (1)$$

The relations (1) are formal in character since, as everyone knows, it is quite likely that both $Z_3(\pi)$ and $Z_3(K)$ are zero. Here we shall explicitly make the following two assumptions:

(A) We first assume that the ratio $Z_3(K)/Z_3(\pi)$ is finite, even though both $Z$'s might be zero. (In the latter case we are assuming the existence of some limiting process which gives a finite value for the ratio.)

(B) Secondly, we assume that the relations

(1) make enough sense to enable us to take their matrix elements between single-particle states, divide, and get

$$\frac{\int d^3x \int d^3y \langle b(q)|[\dot{\pi}_+(\vec{x},t),\pi_-(\vec{y},t)]|b(q')\rangle}{\int d^3x \int d^3y \langle b(q)|[\dot{K}_+(\vec{x},t),K_-(\vec{y},t)]|b(q')\rangle} = \frac{Z_3(K)}{Z_3(\pi)}, \tag{2}$$

where $|b(q)\rangle$ is a single-particle state with four-momentum $q$. For simplicity we shall initially take "$b$" to be a pseudoscalar particle. In writing (2), we have cancelled out on the right-hand side the ratio of two volumes coming from one of the space integrations; i.e., we have cancelled out $(2\pi)^3\delta(\vec{0})$ from numerator and denominator.[4]

One can now apply the methods of Adler and Weisberger to the integrals on the left-hand side of (2). If one does this formally, one obtains the ratio of two divergent sum rules. To give any meaning to such a ratio, one has to introduce a cutoff and interchange the order of the division with the limit. There are several ways in which one can introduce such a cutoff. We choose the following one: We formally rewrite the equal-time commutator as a limit of nonequal-time commutators,

$$[\dot{\pi}(\vec{x},t),\pi(\vec{y},t)] = \int_{-\infty}^{+\infty} dt'\,\delta(t'-t)[\dot{\pi}(\vec{x},t'),\pi(\vec{y},t)] = \lim_{\alpha\to\infty}\int_{-\infty}^{+\infty} dt' \frac{\alpha}{\sqrt{\pi}} \exp[-\alpha^2(t'-t)^2][\dot{\pi}(\vec{x},t'),\pi(\vec{y},t)]. \tag{3}$$

We then rewrite (2) as

$$\frac{\lim_{\Lambda_1\to\infty}\int d^3x\int d^3y\int dt'\,\Lambda_1 \exp\left[-\frac{\Lambda_1^2(t'-t)^2}{q_0^2}\right]\langle b(q)|[\dot{\pi}_+(\vec{x},t'),\pi_-(\vec{y},t)]|b(q')\rangle}{\lim_{\Lambda_2\to\infty}\int d^3x\int d^3y\int dt'\,\Lambda_2 \exp\left[-\frac{\Lambda_2^2(t'-t)^2}{q_0^2}\right]\langle b(q)|[\dot{K}_+(\vec{x},t'),K_-(\vec{y},t)]|b(q')\rangle} = \frac{Z_3(K)}{Z_3(\pi)}, \tag{4}$$

where for later convenience we have set $\alpha = \Lambda/q_0$. The quantity $\Lambda$ has now dimensions of a mass squared. We write

$$\Lambda_1 = \lambda M_{\pi b}^2, \quad \Lambda_2 = \lambda M_{Kb}^2, \tag{5}$$

where $\lambda$ is now a dimensionless parameter and $M_{\pi b}^2$ and $M_{Kb}^2$ are unknown masses which set the scale for the limiting process in the numerator and denominator.

Finally, we make our third assumption: (C) We assume that in (4) we can interchange the order of the division and the limit process. Using this assumption and (5) we write

$$\frac{m_\pi^4 Z_3(K)}{m_K^4 Z_3(\pi)} = \lim_{\lambda\to\infty}\left[\frac{q_0^3 m_\pi^4 M_{\pi b}^2 \int d^3x\int d^3y\int dt' \exp\left(-\frac{\lambda^2 M_{\pi b}^4 (t'-t)^2}{q_0^2}\right)\langle b(q)|[\dot{\pi}_+(\vec{x},t'),\pi_-(\vec{y},t)]|b(q')\rangle}{q_0^3 m_K^4 M_{Kb}^2 \int d^3x\int d^3y\int dt' \exp\left(-\frac{\lambda^2 M_{Kb}^4 (t'-t)^2}{q_0^2}\right)\langle b(q)|[\dot{K}_+(\vec{x},t'),K_-(\vec{y},t)]|b(q')\rangle}\right], \tag{6}$$

where again for later convenience we have multiplied the numerators by $m_\pi^4 q_0^3$ and denominators by $m_K^4 q_0^3$. We insert a complete set of states in both the numerator and denominator, and follow closely the notation of Ref. 2, obtaining

$$\frac{m_\pi^4 Z_3(K)}{m_K^4 Z_3(\pi)}$$

$$= \lim_{\lambda\to\infty}\left[\frac{m_\pi^4 \int_{M_b+m_\pi}^{\infty} dW \sum_j^{\text{int}} \delta(W-M_j)\exp\left\{-\left[\frac{(q_0-q_{j0})q_0}{2\lambda M_{\pi b}^2}\right]^2\right\}\frac{M_b M_j}{q_{j0}q_0}(q_0-q_{j0})q_0^3[|F_j^-(\pi)|^2+|F_j^+(\pi)|^2]}{m_K^4 \int_{M_b+m_K}^{\infty} dW \sum_j^{\text{int}} \delta(W-M_j)\exp\left\{-\left[\frac{(q_0-q_{j0})q_0}{2\lambda M_{Kb}^2}\right]^2\right\}\frac{M_b M_j}{q_{j0}q_0}(q_0-q_{j0})q_0^3[|F_j^-(K)|^2+|F_j^+(K)|^2]}\right], \tag{7}$$

where

$$\langle j|\pi^{\pm}(0)|b(q)\rangle=\left(\frac{M_b}{q_0}\right)^{1/2}\left(\frac{M_j}{q_{j0}}\right)^{1/2}F_j^{\pm}(\pi),\quad \langle j|K^{\pm}(0)|b(q)\rangle=\left(\frac{M_b}{q_0}\right)^{1/2}\left(\frac{M_j}{q_{j0}}\right)^{1/2}F_j^{\pm}(K),\tag{8}$$

and

$$q_{j0}=(q_0^{\,0}+M_j^{\,2}-M_b^{\,2})^{1/2};\quad \vec{q}_{j0}=\vec{q}.\tag{9}$$

The sum $\sum_j^{\rm int}$ also goes over all the internal variables of the system $j$ and $M_j$ denotes the invariant mass of the system $j$.

For finite $\lambda$ the expressions on the right-hand side of (7) are convergent for any $q_0$. We can now take the limit $q_0\to\infty$ in both the numerator and denominator and obtain

$$\frac{m_\pi^{\,4}Z_3(K)}{m_K^{\,4}Z_3(\pi)}=\lim_{\lambda\to\infty}\left[\frac{\int_{M_b+m_\pi}^{\infty}dW\exp\left[-\left(\frac{W^2-M_b^{\,2}}{4\lambda M_{\pi b}^{\,2}}\right)^2\right]W(W^2-M_b^{\,2})^2[\sigma_{\pi^+b}^{\,0}(W)+\sigma_{\pi^-b}^{\,0}(W)]}{\int_{M_b+m_K}^{\infty}dW\exp\left[-\left(\frac{W^2-M_b^{\,2}}{4\lambda M_{Kb}^{\,2}}\right)^2\right]W(W^2-M_b^{\,2})^2[\sigma_{K^+b}^{\,0}(W)+\sigma_{K^-b}^{\,0}(W)]}\right],\tag{10}$$

where $\sigma_{\pi b}^{\,0}$ is the off-mass-shell total cross section for $\pi b$ scattering with the mass of the external pions set equal to zero. We might remark here that the factors $m_\pi^{\,4}$ and $m_K^{\,4}$ are absorbed on the right since they are needed to relate the $F_j$'s to the total cross sections. For details see Ref. 2.

Using the Pomeranchuk theorems $\sigma(\infty)\to$ constant, and $\sigma_{\pi^+b}(\infty)=\sigma_{\pi^-b}(\infty)$, we obtain from (10)

$$\frac{m_\pi^{\,4}Z_3(K)}{m_K^{\,4}Z_3(\pi)}=\left(\frac{M_{\pi b}}{M_{Kb}}\right)^6\frac{\sigma_{\pi b}^{\,0}(\infty)}{\sigma_{Kb}^{\,0}(\infty)}.\tag{11}$$

However, instead of starting with (2), we could have considered the ratio

$$\frac{\int d^3x\int d^3y\langle b(q)|[\dot{\pi}_+(\vec{x},t),\pi_-(\vec{y},t)]|b(q')\rangle}{\int d^3x\int d^3y\langle c(q)|[\dot{\pi}_+(\vec{x},t),\pi_-(\vec{y},t)]|c(q')\rangle}=1.\tag{12}$$

Then following similar steps as before we would obtain

$$\left(\frac{M_{\pi b}}{M_{\pi c}}\right)^6\frac{\sigma_{\pi b}^{\,0}(\infty)}{\sigma_{\pi c}^{\,0}(\infty)}=1.\tag{13}$$

We now let $b$ in (11) and (13) be a pion and $c$ be a kaon, and take the ratio of (11) to (13) to get

$$\frac{m_\pi^{\,4}Z_3(K)}{m_K^{\,4}Z_3(\pi)}=\left(\frac{M_{\pi K}}{M_{K\pi}}\right)^6\frac{\sigma_{\pi K}^{\,0}(\infty)}{\sigma_{K\pi}^{\,0}(\infty)}.\tag{14}$$

We recall that when we write $\sigma_{ab}^{\,0}$, the first particle in the subscript has external four-momentum whose square is zero. Similarly, $M_{\pi K}$ is the scale for the cutoff for the process with the external pions off the mass shell, and $M_{K\pi}$ with external kaon off the mass shell. It is reasonable to assume that at high energies the off-mass-shell corrections are small for both the $K$ and $\pi$. This will give us, as $W\to\infty$,

$$\sigma_{\pi K}(W)\cong\sigma_{\pi K}^{\,0}(W)\cong\sigma_{K\pi}^{\,0}(W).\tag{15}$$

It will also imply that $M_{\pi K}\cong M_{K\pi}$. Thus our final result is obtained:

$$\frac{Z_3(K)}{Z_3(\pi)}\cong\frac{m_K^{\,4}}{m_\pi^{\,4}}.\tag{16}$$

We conclude by making a few remarks about the result we have obtained. First of all it is clear that with the present status of our knowledge of field theory, we have no enlightening remarks to make about the validity of assumptions (A), (B), and (C). We just accept them for the moment. Secondly, the cutoff procedure we have used is not the only one available.

We could, for example, have cut off the sum over states at a mass $M_j = \lambda M_{\pi b}$ in the numerator and $M_j = \lambda M_{Kb}$ in the denominator and not used the Gaussian cutoff. This would have led to truncated integrals in (10) and to the same final result as $\lambda \to \infty$. We have certainly not proved that all cutoff procedures give the same answer. However, if the $Z$'s are nonzero it is hard to see how a different result could be made to match with the ratios considered in this paper. For the case in which both $Z$'s are zero, one could look at above arguments as giving at least two limiting processes which define the ratio $Z_3(K)/Z_3(\pi)$.

Other than (A), (B), and (C), and assuming the uniform form (5) for the cutoffs, our only assumption left is the physical statement that at high energies the off-mass-shell corrections for the $\pi$ and the $K$ are small. This leads us also to the conclusion that the scales $M_{K\pi}$ and $M_{\pi K}$ are equal, which is certainly necessary if (15) holds. We point out further that since we only go off the mass shell in the external particles, the intermediate states $|j\rangle$, all physical, that appear in the unitarity sum for $\sigma_{\pi K^0}$ are identical with the intermediate states for $\sigma_{K\pi^0}$. This gives added reason for setting $M_{\pi K} = M_{K\pi}$.

Finally, we point out that one could apply the same reasoning to spin-$\frac{1}{2}$ particles. Carrying out a similar analysis for example for the $\Lambda$ hyperon and the neutron as we did for $\pi$ and $K$, we get

$$\frac{Z_2(\Lambda)}{Z_2(n)} = \frac{m_\Lambda^2}{m_n^2}. \tag{17}$$

The reason for the different powers in (16) and (17) comes from the fact that in going from the spin-$\frac{1}{2}$ field to its source we use the Dirac operator instead of the Klein-Gordon operator as before. We remark, however, that in this case one feels more uneasy about the off-mass-shell corrections; first, because the masses are larger, and secondly because the off-mass-shell continuation is not done only through functions analytic in the external mass variable, but we have also to use zero-mass spinors. The method considered in this paper cannot determine ratios of $Z_3/Z_2$. For if we take ratios of matrix elements of commutators of spin-zero fields and those of anticommutators of spin-$\frac{1}{2}$ fields, we have to multiply numerator and denominator by different powers of $q_0$ and end up with sum rules that diverge with different powers.

We close by making two remarks about the relation of our result to internal symmetries in general and SU(3) in particular. First, without any internal symmetry it is practically impossible to check the ratio in (16) experimentally. Secondly, if we only take ratios for particles belonging to the same multiplet of an internal symmetry group, the assumption we have made can be tested, as will be shown in detail in a forthcoming paper.

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