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MAGNETIZATION-DENSITY-WAVE INSTABILITY OF A QUANTUM PLASMA IN THE RANDOM-PHASE APPROXIMATION*

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There has been considerable interest recently in the effect of magnetic interactions among the conduction electrons in metals on the de Haas–van Alphen oscillations of the magnetic susceptibility.[1-4] Shoenberg[1] made the conjecture that the correct oscillatory magnetization could be obtained by substituting B for H in the standard expression[5] for the magnetization $M_0(H)$, in which all interaction effects are neglected. Pippard[2] has shown that this procedure leads to a many-valued function $M(H)$, and he has investigated the thermodynamic behavior of the system as a function of H for this situation. Condon[4] has suggested that under appropriate experimental conditions the magnetization within the material will be nonuniform, and that the resulting situation can be described in terms of diamagnetic domains.

The object of the present note is to point out that within the framework of the random-phase approximation, the system will exhibit an instability by spontaneously supporting a spatially nonuniform magnetization of the form $\vec{M}(\vec{r},\vec{H}) = \vec{m}_0(\vec{H}) + \vec{m}_1(\vec{H})\exp(-i\vec{q}\cdot\vec{r}_\perp)$ for that portion of the $M(H)$-vs-H curve for which $(dM/dH) < -\frac{1}{4}\pi$. In the expression given above $\vec{r}_\perp$ is the component of the position vector normal to the dc magnetic field, and $\vec{q}$, the wave vector of the spatially varying magnetization, is a uniquely determined function of H. The values of $\vec{m}_0$ and $\vec{m}_1$ can be obtained from the theory by simple intuitive arguments. We refer to this state with spatially varying magnetization as a magnetization-density-wave (MDW) state. It is apparent that this description of the system is related to Condon's idea of diamagnetic domains, although the exact connection between the MDW state and diamagnetic domains is not completely clear at present.

For simplicity we shall restrict our consideration to an idealized system consisting of N free electrons of charge $-e$ and effective mass m contained in a volume Ω. A dc magnetic field of field strength $\vec{H}_0$ inside the sample is applied parallel to the z axis. We shall follow an approach described in an earlier publication[6] and assume that $\vec{B}_0$, the magnetic induction inside the material, is related to $\vec{H}_0$ by the equation

$$\vec{H}_0 = \mu^{-1}(|\vec{B}_0|)\vec{B}_0. \tag{1}$$

The Hamiltonian for a single electron inside the sample is

$$\mathcal{H}_0 = (1/2m)[\vec{p} + (e/c)\vec{A}_0]^2, \tag{2}$$

where $\vec{A}_0$, the vector potential associated with $\vec{B}_0$, is taken to be $\vec{A}_0 = (0, B_0 x, 0)$. The eigenfunctions and eigenvalues of $\mathcal{H}_0$ are the well-known Landau levels:

$$\mathcal{H}_0|\nu\rangle = \mathcal{H}_0|nk_yk_z\rangle = E_n(k_z)|nk_yk_z\rangle,$$

$$E_\nu = E_n(k_z) = \hbar\omega_c(n+\tfrac{1}{2}) + \hbar^2k_z^2/2m. \tag{3}$$

Here $\omega_c = eB_0/mc$; the quantum number n can be any non-negative integer, and the allowed values of k_y and k_z are determined by applying standard periodic boundary conditions. An in-

finitesimal disturbance in the form of a time-independent magnetic field $\vec{H}_1$ is introduced inside the material. This disturbance induces a response in the system, and the total perturbation acting on any electron is the self-consistent field given by the sum of the magnetic field strength $\vec{H}_1$ and the field set up by the response of all the other electrons in the system to the self-consistent field. Clearly if $\vec{H}_1$ and $\vec{B}_1$ are independent of position, we could replace $\vec{H}_0$ by $\vec{H}_0+\vec{H}_1$ and $\vec{B}_0$ by $\vec{B}_0+\vec{B}_1$ in Eq. (1) and obtain

$$H_1/B_1=\mu^{-1}(|B_0|), \tag{4}$$

for $\vec{H}_1$ normal to $\vec{H}_0$, or

$$\frac{dH}{dB}=\frac{H_1}{B_1}=\mu^{-1}(|B_0|)+B_0\frac{d\mu^{-1}(|B_0|)}{dB_0}, \tag{5}$$

for $\vec{H}_1$ parallel to $\vec{H}_0$. In the present analysis we allow $\vec{H}_1$ to be an arbitrary function of position, and, in the spirit of the random-phase approximation, obtain each Fourier component of the response of the system in terms of the corresponding Fourier component of the self-consistent field.

For simplicity we shall consider H_1 to be of the form $H_1(\vec{r})=H_1(q)e^{-iqy}$. The Hamiltonian for a single electron in the presence of $\vec{B}_0$ and the self-consistent field is then given by

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1, \tag{6}$$

where

$$\mathcal{H}_1=(e/c)(\vec{v}\cdot\vec{A}+\vec{A}\cdot\vec{v}). \tag{7}$$

In Eq. (7) $\vec{v}$ is the velocity of an electron in the presence of $\vec{B}_0$, and $\vec{A}$ is the vector potential of the self-consistent field. In writing down Eq. (7), we have neglected terms of higher order than linear in $\vec{A}$, and we have chosen a gauge such that the scalar potential of the self-consistent field vanishes. We are interested in the diamagnetic response of the system, so for simplicity we have omitted the effect of the electron spin.

We solve the linearized equation of motion of the single-particle density matrix and obtain a relation between matrix elements of $\mathcal{H}_1$ and matrix elements of ρ_1, the deviation of the density matrix ρ from its value ρ_0 in the absence of the self-consistent field. From this relation we can obtain the following expression for the Fourier components of the induced current density[6,7]:

$$j_i(q,\omega)=-(\omega_p{}^2/4\pi c)(\delta_{ij}+I_{ij})A_j(q,\omega), \tag{8}$$

where summation over repeated subscripts is implied, and where

$$I_{ij}=\frac{m}{N}\sum_{\nu\nu'}[f_0(E_{\nu'})-f_0(E_\nu)](E_{\nu'}-E_\nu-\hbar\omega)^{-1}\times\langle\nu'|V_i(q)|\nu\rangle\langle\nu'|V_j(q)|\nu\rangle^*. \tag{9}$$

Here $f_0(E)$ is the Fermi distribution function and the operator $\vec{V}(q)$ is given by

$$\vec{V}(q)=\tfrac{1}{2}e^{i\vec{q}\cdot\vec{r}}\vec{v}+\tfrac{1}{2}\vec{v}e^{i\vec{q}\cdot\vec{r}}. \tag{10}$$

All the nonvanishing matrix elements of $\vec{V}(q)$ are evaluated in Refs. 6 and 7. In the present case the self-consistent field is time independent, so that we need only the zero-frequency limit of Eqs. (8) and (9). Now Maxwell's equations relate the magnetization to the induced current density:

$$\nabla\times\vec{M}=c^{-1}\vec{j}. \tag{11}$$

By considering the case where $\vec{H}_1$ is parallel to $\vec{H}_0$, we obtain the following relation between $H_1(q)$ and $B_1(q)$:

$$\frac{H_1(q)}{B_1(q)}=\frac{dH(q)}{dB(q)}=1+\frac{\omega_p{}^2}{c^2q^2}[1+I_{xx}(q)]. \tag{12}$$

In this equation I_{xx} is a component of the tensor I evaluated at zero frequency. By expanding[6] the matrix elements of $\vec{V}(q)$ in powers of qv_F/ω_c, we obtain the result

$$1+I_{xx}=\frac{q^2}{m\omega_c{}^2}\Big\{\langle\epsilon_\perp{}^2\rangle'+2\langle\epsilon_\perp\rangle-\tfrac{1}{2}\frac{q^2}{m\omega_c{}^2}[\langle\epsilon_\perp{}^3\rangle'+3\langle\epsilon_\perp{}^2\rangle+(\hbar^2\omega_c{}^2/4)(\langle\epsilon_\perp\rangle'+1)]\Big\}. \tag{13}$$

In this equation we have introduced the symbol $\epsilon_\perp=\hbar\omega_c(n+\frac{1}{2})$. Furthermore, the symbols $\langle\Lambda\rangle$ and $\langle\Lambda\rangle'$ denote the following averages:

$$\langle\Lambda\rangle=N^{-1}\textstyle\sum_\nu f_0(E_\nu)\Lambda,$$

$$\langle\Lambda\rangle'=-N^{-1}\textstyle\sum_\nu(\partial f_0(E_\nu)/\partial\xi)\Lambda, \tag{14}$$

where ξ is the chemical potential. The averages appearing in Eq. (13) can be evaluated in terms of oscillatory functions by using the Poisson sum formula in a way described by de Freitas, Quinn, and Rodriguez.[8] By retaining only the leading oscillatory terms in the expressions for $\langle\epsilon_\perp{}^n\rangle$ and $\langle\epsilon_\perp{}^n\rangle'$, we obtain the

result

$$\frac{dM(q)}{dH(q)} = -\frac{1}{4\pi}\left\{1-\left[1-\frac{3\omega_p{}^2 v_F{}^2}{4c^2\omega_c{}^2}\delta_1\left(1-\frac{q^2 v_F{}^2}{4\omega_c{}^2}\right)\right]^{-1}\right\}. \quad (15)$$

Here $M(q)$ is the qth Fourier component of the magnetization and $H(q)$ is the corresponding Fourier component of H_1. The Fermi velocity and electron plasma frequency are denoted by v_F and ω_p, respectively. The symbol δ_1 stands for the oscillatory function

$$\delta_1 = \pi^2 \frac{kT}{\zeta_0}\left(\frac{2\zeta_0}{\hbar\omega_c}\right)^{1/2} \sum_{r=1}^{\infty} \frac{(-1)^r r^{1/2} \cos(\pi g r m_0/m)\cos(2\pi r\zeta_0/\hbar\omega_c - \frac{1}{4}\pi)}{\sinh(2\pi^2 rkT/\hbar\omega_c)}, \quad (16)$$

where ζ_0 is the chemical potential in the absence of the dc magnetic field, g is the electron g factor, and m_0 is the mass of a free electron. The effect of the electron spin on the diamagnetism has been included for the sake of completeness; it can be omitted by setting g equal to zero. If we take the limit as q approaches zero, Eq. (15) is equivalent to the derivative with respect to H of the function $M(H) = M_0(H + 4\pi M)$ discussed by Pippard[2] (actually Pippard discusses the case where only the first harmonic, i.e., the $r = 1$ term in δ_1, is of importance). In Fig. 1 we plot $M(H)$ vs H following Pippard. At points B and D it is apparent that dM/dH diverges; throughout the region between B and D, $dM/dH < -1/4\pi$. It is not difficult to see that in this situation $dM(q)/dH(q)$ will diverge for the value of q given by

$$q = \pm\frac{2\omega_c}{v_F}\left(1-\frac{4}{3}\frac{c^2\omega_c{}^2}{v_F{}^2\omega_p{}^2}\delta_1{}^{-1}\right)^{1/2}. \quad (17)$$

The divergence of $dM(q)/dH(q)$ implies that even for $H(q)$ equal to zero it is possible to have a finite spatially oscillating magnetization $M(q)$. The divergence of $dM(q)/dH(q)$ is equivalent to stating that $\nu(q) = 0$, where $H(q) = \nu(q)B(q)$.

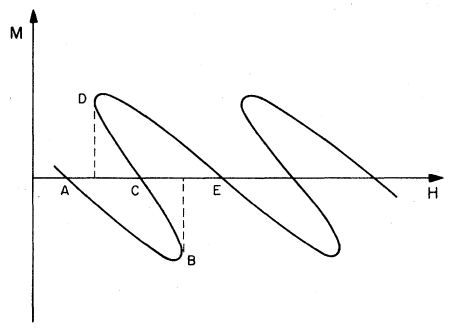

FIG. 1. The magnetization $M(H)$ vs H, the intensity of the magnetic field inside the sample, for a case when magnetic interactions are important.

This situation is somewhat analogous to the familiar situation of an ac electric field in a metal for which $D(\omega) = \epsilon(\omega)E(\omega)$. The condition $\epsilon(\omega) = 0$, the vanishing of the dielectric constant, allows a solution of finite $E(\omega)$ even when there is no driving term $D(\omega)$; this is just the familiar condition for plasma oscillations. In the present study we are interested in the case where H is uniform (at least when depolarization effects are neglected), so that $H(q)$ must vanish for q different from zero. However, we have just shown that a state can still exist for which the magnetization is an oscillatory function of position of the form

$$M = m_0 + m_1(e^{iqy} + e^{-iqy}). \quad (18)$$

where q is given by Eq. (17). Actually q, being a difference between two wave vectors k_y, must satisfy periodic boundary conditions, so that not all values given by Eq. (17) are allowed. The allowed values are the solutions of Eq. (17) which equal $2\pi/L$ times an integer, where L is the length of the sample in the y direction. We can obtain the values of m_0 and m_1 by noting that at $\cos(qy) = \pm 1$, $M(H, y)$ has a maximum or minimum. Because the wavelength of the oscillatory magnetization is very large, we expect these two values to correspond to the two stable solutions $M(H)$ for the given H. If we call these solutions M_+ and M_-, then

$$m_0 = \tfrac{1}{2}(M_+ + M_-),$$

$$m_1 = \tfrac{1}{2}(M_+ - M_-). \quad (19)$$

Thus the present analysis predicts that for $H_D < H < H_B$ (where H_D and H_B are the values of H at points D and B of Fig. 1), a MDW state with magnetization described by Eqs. (17)-(19) can exist. We have found this state by studying the relation between H and B, the magnetic field intensity and magnetic induction inside

the material. To describe the thermodynamic behavior of a real system, H or B must be related to the field outside the sample produced by the external magnet. Thus a number of factors which have been omitted from the present analysis (e.g., boundary conditions, depolarization effects, etc.) will be important. Discussion of real systems will be taken up in future work.

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RADIO-FREQUENCY RESISTANCE IN THE MIXED STATE FOR SUBCRITICAL CURRENTS

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The dc resistivity in the superconducting mixed state is essentially zero unless $|\vec{J}\times\vec{H}_0| \geq \alpha_c$, where $\vec{J}$ is the transport current density, $\vec{H}_0$ is the magnetic field perpendicular to a thin plate sample, and α_c is a measure of the strength of the flux pinning defects. For larger currents the flux tubes flow over the pinning centers giving a dynamic resistivity $\rho_f = \rho_n H_0/H_{c2}$ where ρ_n is the normal resistivity.[1] For an "ideal" material (no pinning centers), $\alpha_c = 0$. At microwave frequencies the resistance of all materials is that of the "ideal" material, even for transport currents several orders of magnitude below the critical value.[2] In this Letter a simple model for the oscillatory motion of flux tubes in the presence of pinning centers is given, and the frequency dependence of the flux-flow resistance is calculated and is shown to be in reasonable agreement with experiments on Pb-In and Nb-Ta alloys. It appears that rf techniques will prove valuable in studying the detailed nature of flux pinning centers in the mixed state.

At fields well above H_{c1} the magnetic energy density required to change the flux-tube lattice constant d is considerably greater than that involved in pinning.[3] The lattice is therefore quite rigid, and, over a distance large compared to d, the flux tubes form a "single crystallite." The size of the crystallites will be determined by the relative strengths of the pinning and magnetic energies. These crystallites are equivalent to the "flux bundles" of Anderson and Kim.[3] If a force (due to a transport current) is exerted on these crystallites, they will displace with respect to the pinning centers. Since the displacement of all of the crystallites is approximately the same, the forces they exert on each other due to the displacement will be small. Following Anderson, we assume that they can slide readily with respect to each other. The equilibrium position and orientation of these crystallites will largely be determined by the potential energy due to the local pinning centers. If there are many pinning centers in each flux-tube crystallite, the potential energy or "pinning potential" will be periodic in the flux-tube lattice constant. It is reasonable to assume that the pinning potential is a fairly smooth function of x, well approximated by $\mathcal{P} = A[1 - \cos(2\pi x/d)]$. The force $\mathcal{P}$ exerts on each flux tube is

$$F = \frac{\partial \mathcal{P}}{\partial x} = -\frac{2\pi A}{d}\sin\frac{2\pi x}{d}.$$