POSITIVITY CONDITIONS FOR THE DENSITY MATRIX OF SPIN-ONE PARTICLES

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The purpose of this Letter is to show that besides the well-known condition $\mathrm{Tr}\rho^2 \leqslant 1$, the spin-one density matrix has to satisfy also the condition $2\,\mathrm{Tr}\rho^3 - 3\,\mathrm{Tr}\rho^2 + 1 \leqslant 0$ which is more restrictive than the first one. This second condition, just as the first one, is shown to be easily expressed in terms of conditions on the multipole parameters.

In the last few years extensive use has been made of the moments method for determination of spin of particles and resonances. Owing to large experimental errors in determination of multipole parameters, one cannot always decide the spin of the studied particle (Ξ, Y^*, Ξ^*). One of the most effective tools to eliminate ambiguities is the theoretical limitation imposed on multipole parameters by the condition $\mathrm{Tr}\rho^2 \leqslant 1$, deduced from the positivity of the density matrix ρ. Up to now it is the only condition one takes into account. In fact, for spin $j > \frac{1}{2}$, further conditions, more restrictive than the usual one, can be obtained from the positivity of ρ. It is hoped that these supplementary theoretical limitations on multipole parameters will be useful to remove some of the remaining ambiguities. General results for any spin will be given in a forthcoming publication. The aim of this Letter is to exhaust the positivity condition in the case of spin-one particles.

The transformation properties of the spin-density operator allows one to write it as a sum of irreducible tensorial operations[1] $T_M{}^L$:

$$\rho = (2j+1) \sum_{L=0}^{2j} \sum_{M=-L}^{+L} (2L+1) t_M{}^{L*} T_M{}^L. \quad (1)$$

The $t_M{}^L$ are the so-called multipole parameters.[2] The general properties of the density operator prescribe some conditions on these parameters. The Hermiticity of ρ implies $t_M{}^L = (-)^M t_{-M}{}^{L*}$, the property $\mathrm{Tr}\rho = 1$ implies $t_0{}^0 = 1$, and the positivity property together with the property $\mathrm{Tr}\rho = 1$ lead to the condition $(2j+1)^{-1} \leqslant \mathrm{Tr}\rho^2 \leqslant 1$ or

$$(2j+1)^{-1} \leqslant (2j+1)^{-1} \sum_{L=0}^{2j} \sum_{M=-L}^{+L} (2L+1) |t_M{}^L|^2 \leqslant 1. \quad (2)$$

(The equality on the right corresponds to a pure state, and the equality on the left corresponds to a completely unpolarized state.)

The spin-one density matrix is a 3×3 matrix. Its characteristic polynomial is

$$\Delta(\lambda) = \lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3). \quad (3)$$

The coefficients c_1, c_2, c_3 are connected with the proper values λ_i (by Newton formulas) and with $\mathrm{Tr}\rho$, $\mathrm{Tr}\rho^2$, $\mathrm{Tr}\rho^3$. It is easy to check the following relations:

$$c_1 = \lambda_1 + \lambda_2 + \lambda_3 = \mathrm{Tr}\rho, \quad (4a)$$

$$c_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = \tfrac{1}{2}[-\mathrm{Tr}\rho^2 + (\mathrm{Tr}\rho)^2], \quad (4b)$$

$$c_3 = \lambda_1\lambda_2\lambda_3 = \tfrac{1}{6}[2\,\mathrm{Tr}\rho^3 - 3\,\mathrm{Tr}\rho^2\,\mathrm{Tr}\rho + (\mathrm{Tr}\rho)^3]. \quad (4c)$$

The matrix ρ will be positive if its proper values are non-negative ($\lambda_i \geqslant 0$). It is obvious from the above relations that a necessary and sufficient condition for ρ to be positive is $c_1 \geqslant 0$, $c_2 \geqslant 0$, $c_3 \geqslant 0$. ρ being a density matrix $c_1 = \mathrm{Tr}\rho = 1$, and the remaining conditions[3] are as follows:

$$-\mathrm{Tr}\rho^2 + 1 \geqslant 0, \quad (5a)$$

$$2\,\mathrm{Tr}\rho^3 - 3\,\mathrm{Tr}\rho^2 + 1 \geqslant 0. \quad (5b)$$

Condition (5a) is nothing but the well-known condition (2) ($\mathrm{Tr}\rho^2 \leqslant 1$). Condition (5b) is not so well known and has never been used, although, as we shall see in a moment, it is more restrictive than condition (5a). To obtain conditions on the multipole parameters, one has to calculate $\mathrm{Tr}\rho^2$ and $\mathrm{Tr}\rho^3$ as functions of the $t_M{}^L$. This can be easily done by application of Racah calculus to the multipole expansion of the matrix ρ. General results including all $t_M{}^L$ are not too complicated; however, to simplify the expressions one can take advantage of some restrictions imposed on $t_M{}^L$ by symmetry prop-

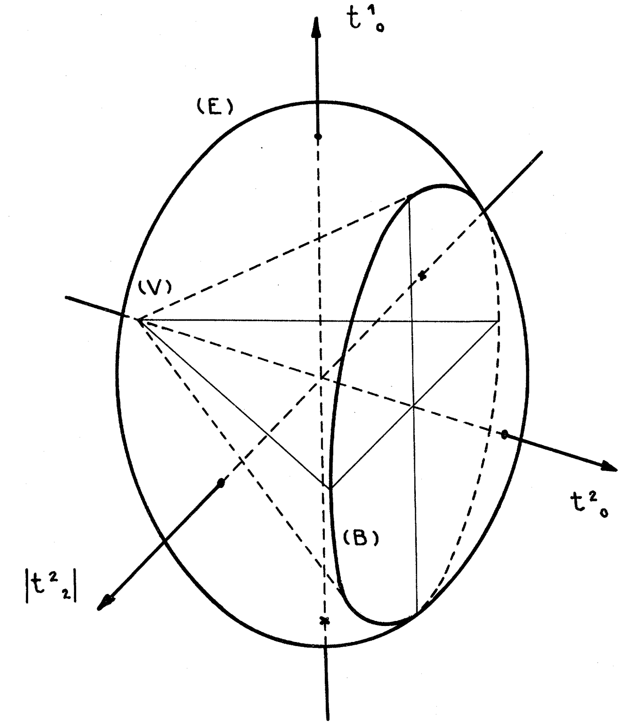

FIG. 1. Domains of variation of the parameters t_M^L.

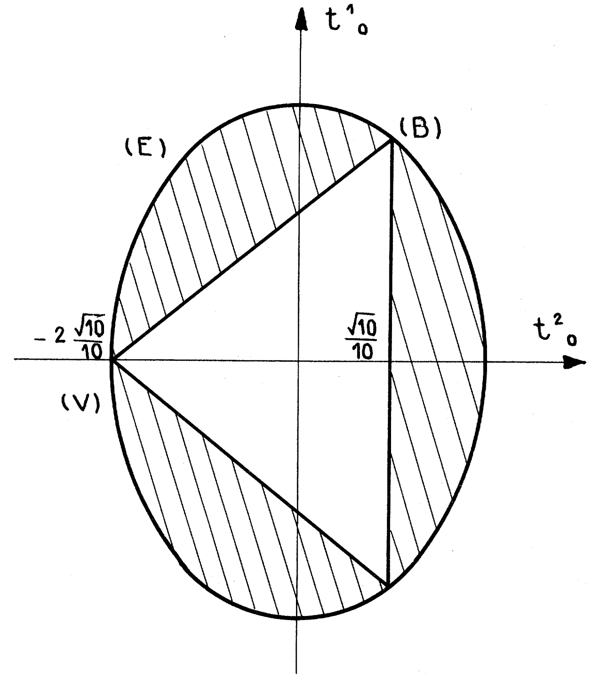

FIG. 2. Domains of variation of t_0^1 and t_0^2 for $|t_2^2| = 0$. The shaded region corresponds to the part of the domain which is ruled out by the supplementary condition.

erties of the production process.[4] If the spin-one particle is produced in a parity-conserving two-body reaction, if we choose the axis of quantization along the normal to the production plane, if the initial particles are unpolarized, and if we average over spin and direction of the other produced particle, then $t_M^L = 0$ for odd M. Thus we remain only with $t_0^0 = 1$, t_0^1, t_0^2, and t_2^2, and we have

$$\mathrm{Tr}\rho^2 = \tfrac{1}{3}[3(t_0^1)^2 + 5(t_0^2)^2 + 10\,|t_2^2|^2 + 1], \tag{6a}$$

$$\mathrm{Tr}\rho^3 = \tfrac{1}{3}\{3(t_0^1)^2[1 + (\tfrac{1}{2}\sqrt{10})t_0^2] + 5(t_0^2)^2[1 - (\tfrac{1}{6}\sqrt{10})t_0^2] + 10\,|t_2^2|^2[1 + (\tfrac{1}{2}\sqrt{10})t_0^2] + \tfrac{1}{3}\}. \tag{6b}$$

The positivity condition (5a) gives

$$3(t_0^1)^2 + 5(t_0^2)^2 + 10\,|t_2^2|^2 - 2 \leq 0. \tag{7}$$

If we represent this in a three-dimensional space, we see that the first positivity condition enforces the representative point whose coordinates are t_0^1, t_0^2, $|t_2^2|$ to be inside an ellipsoid (E) (Fig. 1). When the representative point is on the surface of the ellipsoid, we have $\mathrm{Tr}\rho^2 = 1$, i.e., we have a pure state. For a representative point inside the ellipsoid, we can introduce a degree of polarization[5] δ $(0 \leq \delta \leq 1)$ defined by

$$\delta^2 = (2j)^{-1}[(2j+1)\,\mathrm{Tr}\rho^2 - 1]. \tag{8}$$

For spin-one particle, we find

$$\mathrm{Tr}\rho^2 = (2\delta^2 + 1)/3. \tag{9}$$

Consequently, the representative points of a given degree of polarization δ are on an ellipsoid similar to ellipsoid (E). Its equation is

$$3(t_0^1)^2 + 5(t_0^2)^2 + 10\,|t_2^2|^2 - 2\delta^2 = 0. \tag{10}$$

Now, the second positivity condition (5b) gives

$$3(t_0^1)^2 - (5/3)(t_0^2 + 2/\sqrt{10})^2 + 10\,|t_2^2|^2 \begin{cases} \leq 0 \text{ for } t_0^2 < 10^{-1/2}, \\ \geq 0 \text{ for } t_0^2 > 10^{-1/2}. \end{cases} \tag{11}$$

This condition enforces the representative point to be inside a frustrum of a cone inscribed in ellipsoid (E) as shown in Fig. 1. The cone has its vertex (V) on the ellipsoid and it intersects the ellipsoid along an ellipse (B). Thus we see that the supplementary condition is more restrictive than the first one. The range of variation of the parameters is reduced, and what is more important, we see that the representative point for a pure state cannot vary over a whole surface as was supposed previously, but has to be either at the vertex (V) of the cone, or on the ellipse (B), i.e., we can only have a pure state for two values of the parameter t_0^2 (see Fig. 2):

$$(V)\quad t_0^2=-2/\sqrt{10};\ \text{then } t_0^1=0,\ \ |t_2^2|=0;$$

$$(B)\quad t_0^2=1/\sqrt{10};\ \text{then } 3(t_0^1)^2+10\,|t_2^2|^2-\tfrac{3}{2}=0. \qquad (12)$$

On the cone the degree of polarization has a simple expression. We have, from (5b), $2\,\mathrm{Tr}\rho^3 -3\,\mathrm{Tr}\rho^2+1=0$, and from (9), $-3\,\mathrm{Tr}\rho^2+1=-2\delta^2$, thus on the cone

$$\delta^2=\mathrm{Tr}\rho^3. \qquad (13)$$

Experimentally one measures t_0^2 and $|t_2^2|$ by means of the angular distribution of decay products of spin-one particles. The parameter t_0^1 cannot be measured in a parity-conserving decay into spin-0 particles [$K^*(891)\rightarrow K+\pi$, $\rho\rightarrow\pi+\pi,\cdots$]; however, formula (11) provides an upper limit on values of t_0^1 and formula (13) an upper limit on the degree of the polarization δ.

We are grateful to Dr. E. de Rafael for bringing this problem to our attention, and to Professor Louis Michel for continuous encouragement.

Note added in proof.—After this Letter was written we learned from Professor R. H. Dalitz that he had given similar results in his lectures at the International School of Physics held at Varenna in July 1964 (lecture notes to be published).

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[1]N. Byers and S. Fenster, Phys. Rev. Letters 11, 52 (1963); C. Henry and E. de Rafael, Ann. Inst. Henri Poincaré 2, 87 (1965); E. de Rafael, thesis (unpublished).

[2]The asterisk (t_M^{L*}) indicates complex conjugation.

[3]These conditions have also been obtained by D. N. Williams (unpublished).

[4]See Ref. 1 and R. H. Dalitz, Ann. Rev. Nucl. Sci. 13, 339 (1963).

[5]U. Fano, National Bureau of Standards Report No. 1214, 1952 (unpublished).