CAN I^2 BE CONSERVED, BUT NOT $\vec{I}$?

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It is usually believed that the strong interactions conserve all three components of isotopic spin, $\vec{I}$. However, in a recent issue of this journal,[1] Vander Velde has suggested that they may, in fact, conserve only I_z and I^2. In a nonrelativistic limit, it is easy to construct models which have this property (e.g., particles interacting through a potential which is polynomial in I_z). The purpose of this note is to show that in relativistic local field theory it is impossible to fulfill Vander Velde's speculation; if I^2 is conserved, $\vec{I}$ is also.

A precise statement[2] of our result is this: We assume there exists a triplet of local Hermitian vector currents, $\vec{j}_\nu(x)$. We define the total isospin vector at time t by

$$\vec{I}(t)=\int d^3x\,\vec{j}_0(\vec{x},t).$$

We assume I^2 is a well-defined operator whose domain includes the usual scattering states (actually, all we need is the vacuum), and which is independent of time. We then show that $\vec{I}$ is independent of time.

Proof: Since I^2 commutes with the Hamiltonian, the vacuum must be an eigenstate of I^2:

$$I^2|0\rangle=\lambda|0\rangle, \tag{2}$$

with λ some finite non-negative number. Thus

$$\lambda=\langle 0|I^2|0\rangle$$

$$=\langle 0|I_x^{\,2}(t)|0\rangle+\langle 0|I_y^{\,2}(t)|0\rangle+\langle 0|I_z^{\,2}(t)|0\rangle, \tag{3}$$

which implies

$$\langle 0|I_x^{\,2}(t)|0\rangle\le\lambda. \tag{4}$$

On the other hand, if

$$\langle 0|j_{0x}(\vec{x},t)\int d^3y\,j_{0x}(\vec{y},t)|0\rangle=a, \tag{5}$$

then the translational invariance of the vacuum implies that a is independent of $\vec{x}$ and t; thus,

$$\langle 0|I_x^{\,2}(t)|0\rangle=\int d^3x\,a. \tag{6}$$

The only way Eqs. (4) and (6) can be consistent is if $a=0$, which implies

$$\langle 0|I_x^{\,2}(t)|0\rangle=\|I_x(t)|0\rangle\|^2=0, \tag{7}$$

or

$$I_x(t)|0\rangle=0. \tag{8}$$

However, it is known[3] that under the stated conditions, this implies

$$(\partial/\partial t)I_x(t)=0. \tag{9}$$

The same arguments apply to the other two components of $\vec{I}$. This completes the proof.

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[1]J. C. Vander Velde, Phys. Rev. Letters 16, 69 (1966).

[2]The proof below is— for the sake of brevity and clarity— sloppy rather than rigorous, and the statement of the theorem given here is not quite careful enough to form the basis of a rigorous proof. The difficulty is that the integral in Eq. (1) does not in general converge; it converges only when evaluated between quasilocal states. Therefore it does not give a direct definition of I^2. If we define $\int_R d^3x$ as the integral over a sphere of radius R, define I^2 as

$$\text{weak}\lim_{R\to\infty}\sum_{i=1}^{3}\int_R d^3x\,j_{0i}(\vec{x},t)\int_R d^3y\,j_{0i}(\vec{y},t),$$

and assume that this limit defines an operator with the stated properties, then we may easily construct a rigorous proof along the lines given in the text.

[3]S. Coleman, "The Invariance of the Vacuum is the Invariance of the World" (to be published). This is where we exploit locality and Lorentz invariance; the entire argument up to this point would be equally valid for nonlocal currents in a nonrelativistic field theory.