IMPROVED VERSION OF OPTICAL EQUIVALENCE THEOREM

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Each of the present formulations[1] of Sudarshan's optical equivalence theorem[2] involves a sequence of "diagonal" operators ρ_M that converge to a desired density operator ρ in Hilbert-Schmidt norm.[3] For various applications, as well as for reasons of completeness, it would be preferable if a "diagonal" operator sequence existed that converged to an arbitrary ρ in the stronger norm characteristic of traceable operators, namely, the trace-class norm, $\| \; \|_1$.[3] This possibility is realized herein, the argument being presented for a single degree of freedom. If $|p,q\rangle \equiv |\alpha\rangle$, $\alpha \equiv (q+ip)/\sqrt{2}$, denotes the usual coherent states of Glauber,[4] then we prove the following:

Theorem: For any trace-class operator T, a sequence of operators of the form

$$T_M = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_M(p,q)\,|p,q\rangle\langle p,q|\,(dp\,dq/2\pi) \quad (1)$$

exists, where $\varphi_M(p,q) \in \mathcal{S}_2$ (Schwartz's space of test functions[5] of two variables of rapid decrease) for $M=1,2,\cdots$, and for which

$$\|T-T_M\|_1 \to 0. \quad (2)$$

Before proceeding to the proof, we indicate an important consequence of this theorem. If B denotes an arbitrary bounded operator and $\|B\|$ denotes its operator norm, then

$$\mathrm{Tr}(TB) = \lim_{M\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_M(p,q)\langle p,q|B|p,q\rangle(dp\,dq/2\pi), \quad (3)$$

since $\varphi_M(p,q) \in \mathcal{S}_2$ and since[3]

$$|\mathrm{Tr}[(T-T_M)B]| \leq \|B\|\cdot\|T-T_M\|_1 \to 0.$$

From a physical point of view, we may state the following: For any state of a system (any density matrix ρ), we can represent the mean $\langle B\rangle \equiv \mathrm{Tr}(\rho B)$ of every bounded operator of unit norm, uniformly (i.e., independent of which B) to an arbitrary accuracy with aid of a "diagonal" weight function (quasiclassical phase-space distribution[2,1]) which is infinitely differentiable and falls off at infinity faster than any inverse power. One could hardly ask for more in applications to quantum optics!

Proof of theorem.—We construct a sequence of "diagonal" operators $\{T_M\}$ such that for any pregiven $\epsilon>0$, there exists an $M=M_\epsilon<\infty$ for which $\|T-T_{M'}\|_1 \leq \epsilon$ for all $M' \geq M_\epsilon$. We approach this construction in three steps. First of all, recall[3] that every trace-class operator T admits the polar decomposition

$$T = \sum_{j=1}^{\infty} \beta_j |\lambda_j\rangle\langle\psi_j|,$$

where $|\lambda_j\rangle$ and $|\psi_j\rangle$ are two complete orthonormal sequences, and where $\beta_j \geq 0$ and

$$\sum_{j=1}^{\infty}\beta_j \equiv \|T\|_1 < \infty.$$

It is well known[3] that the sequence

$$T^J \equiv \sum_{j=1}^{J}\beta_j|\lambda_j\rangle\langle\psi_j|$$

converges to T in trace-class norm as $J\to\infty$. Let us, therefore, choose $J=J(\epsilon)<\infty$ as the least integer such that $\|T-T^J\|_1 \leq \epsilon/3$. For the next step, define

$$T_N^J = \sum_{j=1}^{J}\beta_j|\lambda_j;N\rangle\langle\psi_j;N|,$$

where, in terms of normalized harmonic oscillator eigenstates $|n\rangle$, we set

$$|\lambda_j;N\rangle = \sum_{n=0}^{N}|n\rangle\langle n|\lambda_j\rangle,$$

$$|\psi_j;N\rangle = \sum_{n=0}^{N}|n\rangle\langle n|\psi_j\rangle,$$

for all $j \leq J$. We choose $N=N(\epsilon)<\infty$ as the least integer such that (for $\|T\|_1>0$)

$$\| |\lambda_j\rangle - |\lambda_j;N\rangle \| + \| |\psi_j\rangle - |\psi_j;N\rangle \| \leq \bar{\epsilon} \equiv \tfrac{1}{2}\{[1+(4\epsilon/3\|T\|_1)]^{1/2}-1\},$$

for all $j \leq J$. In consequence,

$$\|T^J - T_N^{\ J}\|_1 = \|\sum_{j=1}^{J} \beta_j\{|\lambda_j\rangle(\langle\psi_j| - \langle\psi_j;N|) + (|\lambda_j\rangle - |\lambda_j;N\rangle)\langle\psi_j;N|\}\|_1$$

$$\leq \sum_{j=1}^{J} \beta_j\{\| |\psi_j\rangle - |\psi_j;N\rangle\| + \| |\lambda_j\rangle - |\lambda_j;N\rangle\| \cdot \| |\psi_j;N\rangle \|\} \leq \sum_{j=1}^{J} \beta_j\{\bar{\epsilon}+\bar{\epsilon}^2\} \leq \epsilon/3.$$

If $U[k,x] \equiv \exp i(kQ-xP)$, $[Q,P]=i$, denotes the unitary Weyl operators, then we have the representation

$$T_N^{\ J} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t_N^{\ J}(k,x) U[k,x](dkdx/2\pi),$$

where[6]

$$t_N^{\ J}(k,x) = \mathrm{Tr}(T_N^{\ J} U^\dagger[k,x]) \equiv \tilde{\varphi}_N^{\ J}(k,x)\langle 0|U^\dagger[k,x]|0\rangle,$$

the latter expression serving to define $\tilde{\varphi}_N^{\ J}(k,x)$. From previous work,[6] the Fourier transform (as a distribution) of $\tilde{\varphi}_N^{\ J}$ yields the Sudarshan "weight function." Since $t_N^{\ J}(k,x) \in \mathcal{S}_2$ for all J and N,[7] and since $\langle 0|U^\dagger[k,x]|0\rangle = \exp[-\frac{1}{4}(k^2+x^2)]$, it follows that $\tilde{\varphi}_N^{\ J}(k,x)$ is C^∞; but, with $N<\infty$, $\tilde{\varphi}_N^{\ J}(k,x)$ cannot have the rapid decrease to belong to $\mathcal{S}_2$. To fix up the rapid decrease (and maintain the differentiability), we may, for example, introduce

$$\tilde{\varphi}_{N,L}^{\ J}(k,x) \equiv \tilde{\varphi}_N^{\ J}(k,x)\exp\{-W_L(k,x)\},$$

where

$$W_L(k,x) = f(x-L) + f(-x-L) + f(k-L) + f(-k-L),$$

and where

$$f(y) \equiv y^4 \exp(-1/y^2), \quad y>0,$$
$$\equiv 0, \quad y \leq 0.$$

With this modification, the function $\tilde{\varphi}_{N,L}^{\ J}(k,x)$, as well as the function

$$t_{N,L}^{\ J}(k,x) \equiv t_N^{\ J}(k,x)\exp\{-W_L(k,x)\},$$

are members of $\mathcal{S}_2$ for all J, N, and L. Moreover, it is straightforward to show, for fixed J and N, that $t_{N,L}^{\ J} \to t_N^{\ J}$ in the topology of Schwartz on $\mathcal{S}_2$, as $L\to\infty$.[5] This is sufficient to imply, for fixed J and N, that $T_{N,L}^{\ J} \to T_N^{\ J}$ in trace-class norm as $L\to\infty$, where

$$T_{N,L}^{\ J} \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t_{N,L}^{\ J}(k,x)U[k,x](dkdx/2\pi).$$

Thus, let us choose $L=L(\epsilon)<\infty$ as the least integer such that, for the previously determined J and N,

$$\|T_N^{\ J} - T_{N,L}^{\ J}\|_1 \leq \epsilon/3.$$

Hence, if we define the sought-for operator sequence by

$$T_M \equiv T_{N(1/M),L(1/M)}^{J(1/M)},$$

i.e., for those special $\epsilon=1/M$, $M=1,2,\cdots$, then it follows from the foregoing, for all $M' \geq M_\epsilon \geq 1/\epsilon$, that

$$\|T-T_{M'}\|_1 \leq 1/M' \leq \epsilon,$$

which establishes the desired convergence. Lastly we note that

$$\tilde{\varphi}_M(k,x) \equiv \tilde{\varphi}_{N(1/M),L(1/M)}^{J(1/M)}(k,x) \in \mathcal{S}_2$$

is the (double) Fourier transform of the weight function $\varphi_M(p, q)$ in the "diagonal" representation (1) of T_M,[6] which establishes that the weight function $\varphi_M(p, q) \in \mathcal{S}_2$ for all M, and concludes our proof.

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[2]E. C. G. Sudarshan, Phys. Rev. Letters 10, 277 (1963); and in Proceedings of the Symposium on Optical Masers (Polytechnic Press, Brooklyn, New York, 1963), p. 45.

[3]Operator and operator-norm properties sufficient for this paper are contained in the independent and readable I. M. Gel'fand and N. Ya. Vilenkin, Generalized Functions, translated by A. Feinstein (Academic Press, New York, 1964), Vol. 4, Chap. I, Sec. 2, p. 26-56.

[4]R. J. Glauber, Phys. Rev. Letters 10, 84 (1963); Phys. Rev. 131, 2766 (1963).

[5]See, for example, I. M. Gel'fand and G. E. Shilov, Generalized Functions, translated by E. Saletan (Academic Press, New York, 1964), Vol. 1, p. 16-18.

[6]J. R. Klauder, J. McKenna, and D. G. Currie, J. Math. Phys. 6, 734 (1965); J. C. T. Poole, to be published.

[7]Implicit from the fact that $\langle n|U^+[k,x]|m\rangle \in \mathcal{S}_2$ for all n, m by direct computation of M. S. Bartlett and J. E. Moyal, Proc. Cambridge Phil. Soc. 45, 545 (1949).

POLARIZATION IN p-p ELASTIC SCATTERING FROM 0.75 TO 2.8 GeV*

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In a recent experiment at the Brookhaven Cosmotron we have measured the polarization parameter in p-p elastic scattering at six proton energies in the range 0.75 to 2.8 GeV. The polarization in p-p scattering has been well determined for proton energies below 750 MeV in many cyclotron experiments, but data are meager at higher energies.[1] The purpose of our measurements was to improve the knowledge of this fundamental parameter in the nucleon-nucleon interaction in an energy range inaccessible to cyclotrons.

The experiment employed a double-scattering technique with scintillation counters as detectors. The experimental arrangement is shown in Fig. 1. The proton beam was extracted from the Cosmotron at the desired energy and focused on a 3-in.-long liquid hydrogen target. Elastic events were selected by requiring a coincidence between counter telescopes S_1S_1' and S_0S_2 which detected both the fast and slow (recoil) protons from an elastic event at the proper kinematic angles. The requirements on coplanarity and relative angle thus imposed on the two protons were so stringent that contamination from inelastic events was negligible at all energies and angles studied. The two telescopes were mounted on rails so that both angles could be varied easily. After passing through S_2 the recoil protons were scattered a second time from a graphite target, and the asymmetry of the doubly-scattered protons was measured by telescopes T_1T_2 and U_1U_2. These telescopes could be interchanged by rotating the entire analyzer about the graphite target so that instrumental asymmetries cancelled out. Thick graphite targets and a poor geometry were used in the analyzer to obtain high efficiency for the second scattering. With this arrangement 0.5 to 3% of the protons entering the graphite scattered into either of the telescopes, and the over-all counting rate was about 20 events per beam pulse of $\approx 4\times10^9$ protons incident on the hydrogen target. Each data point required about 60 min of running time.

Important accidental rates were monitored constantly and were always small enough that corrections to the measured asymmetries were unnecessary. The target-empty rate was found to be negligible in all cases. Checks were also made to insure that the external proton beam was unpolarized. The most serious source of potential systematic error in experiments of this type is the possibility of a misalignment of the axis of rotation of the analyzer relative to the proton beam entering it. In this experiment spark chambers were used just before