USE OF UNITARITY IN PROVING POMERANCHUK'S THEOREM ON CROSS SECTIONS AT HIGH ENERGIES*

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Pomeranchuk[1] suggested in 1956 that total cross sections for particle-particle and for particle-antiparticle scattering become equal in the limit of high energy. There have since been a number of proofs of the Pomeranchuk theorem[2-8] and its generalizations. All these proofs make assumptions about analyticity, crossing symmetry, and temperedness, which have to a large extent been established from the axioms of quantum field theory. In addition, two assumptions are made which have not been shown to have the same support from axiomatic field theory. These two additional assumptions have been stated in many different ways[2-8] depending on which generalization or what degree of rigor the author required.

(1) The first extra assumption limits the rate of change of the difference between particle and antiparticle cross sections. In a typical example,

$$\sigma_+(E)-\sigma_-(E)=C\varphi(E) \tag{1}$$

is assumed to change no faster than a power of a logarithm of the laboratory energy E.

(2) The second additional assumption limits the rate of change of the modulus of the difference between the corresponding amplitudes. For example, it is assumed that

$$\frac{|F_+(E)-F_-(E)|}{|E\ln E|\,|\varphi(E)|}\to 0 \text{ as } E\to\infty. \tag{2}$$

It is the purpose of this Letter (a) to note first the essential role of the above assumptions in proofs of the Pomeranchuk theorem; (b) to observe that if one considers only forward scattering there is no obvious reason why (2) should be valid except when $\varphi(E)$ is chosen to increase so fast that the Froissart bound[9] is violated; (c) to note that within certain limits on the rate of change of the total cross sections $\sigma_+(E)$ and $\sigma_-(E)$, the assumption (2) can be replaced by unitarity.

The full generality imposed by possible oscillations might obscure my main points so I will, in this Letter, illustrate them by assuming simple functions only. I consider first the proof of the Pomeranchuk theorem from (1) and (2), together with analyticity, crossing, and temperedness. The amplitude

$$g(E)=[F_+(E)-F_-(E)]/(E^2-\mu^2)^{1/2} \tag{3}$$

satisfies the condition[5,6]

$$g(E+i0)=g(-E+i0). \tag{4}$$

Using temperedness and analyticity, it is possible to establish conditions on the rate of change of the real part of $g(E)$ when its imaginary part is given. Thus from analytic function theory,[5,6] if, as $E\to\infty$,

$$\operatorname{Im}\{g(E)\}\sim[\sigma_+(E)-\sigma_-(E)]\sim C(\ln E)^m, \tag{5}$$

then

$$\operatorname{Re}\{g(E)\}\sim -C'(\ln E)^{m+1}, \tag{6}$$

where

$$C'=2C/\pi(m+1).$$

The conclusion (6) contradicts assumption (2) unless $C=0$. This proves the theorem.

We see that the proof involves the use of an assumption (2) which precludes the rate of growth of the real part of $g(E)$ that is a consequence of analytic function theory when its imaginary part behaves as assumed. This use of an assumption of the type (2) is a general feature of all existing proofs of the Pomeranchuk theorem,[2-8] and although its presence is sometimes clandestine it is normally overt. It is therefore of great importance to analyze the validity or justification of the assumption (2). In our example it is equivalent to the assumption that the amplitudes $F_+(E)$ and $F_-(E)$ should not become dominantly real at high energies. Although this has been suggested to be a "good" physical assumption,[10] it cannot be justified in general, and should only be regarded as a physical assumption when it can be confirmed by unitarity or some other physical condition.

Unitarity requires that

$$\sigma_\pm(\text{total})\geq\sigma_\pm(\text{elastic}). \tag{7}$$

A lower bound on σ(elastic) can be derived by using temperedness in E together with holomorphy in t for $\cos\theta$ inside the largest ellipse compatible with perturbation theory, and using analyticity and crossing in E.[11] Temperedness in E and holomorphy in t establish that no more than L terms contribute to the partial wave series for large E,[9]

$$F_{\pm}(E,t)=\sum_{0}^{\infty}(2l+1)f_l(E)P_l(1+t/E), \tag{8}$$

where $f_l(E)$ is negligible unless $l<L$ where

$$L^2 \leqslant E(\ln E)^2. \tag{9}$$

It is possible from this, using unitarity, to bound not only $F_\pm(E,0)$ giving the Froissart bound, but also to bound every derivative of $F_\pm(E,t)$ at $t=0$, if E is large enough.[11] In particular, one can show that $\mathrm{Im}F_\pm(E,t)$ cannot decrease (as t goes negative into the physical region) faster than $c_1 \mathrm{Im}F(E,0)t(\ln E)^2$. Using crossing symmetry in E, a similar result can be shown to be true for $[\mathrm{Re}F_+(E,t)\pm\mathrm{Re}F_-(E,t)]$ and hence for $\mathrm{Re}F_\pm(E,t)$.

In order to prove the Pomeranchuk theorem, we assume that either or both σ_+ and σ_- behave like $C_\pm(\ln E)^m$ for large E, and their difference (5) has $C\neq 0$. From (6) the real part of the difference F_+-F_- exceeds the imaginary part by a factor $\ln E$. It follows from $C\neq 0$ that the larger of $|\mathrm{Re}F_\pm(E,0)|$ will exceed the corresponding $\mathrm{Im}F_\pm(E,0)$ also by a factor $\ln E$. We can use this result plus the bound on the rate of change of $\mathrm{Re}F_\pm(E,t)$ to give a lower bound on $\sigma_\pm$(elastic), using

$$\sigma_{\pm}(\text{elastic}) > \int_{-t_0}^{0} dt \left|\frac{F_\pm(E,t)}{E}\right|^2 > \int_{-t_0}^{0} dt \left|\frac{\mathrm{Re}F_\pm(E,t)}{E}\right|^2, \tag{10}$$

where t_0 is a constant less than the distance from $t=0$ to the nearest singularity of $F_\pm(E,t)$. We now use unitarity in the form

$$\sigma_{\pm}(\text{total})=\frac{\mathrm{Im}F_\pm(E,0)}{E}\geqslant\sigma_{\pm}(\text{elastic}). \tag{11}$$

We use the larger of $|\mathrm{Re}F_\pm(E,0)|$ in (10) since this is dominated by $|\mathrm{Re}F_+(E,0)-\mathrm{Re}F_-(E,0)|$. Then, on substituting $(\ln E)^m$ for $\sigma_\pm$, and $C(\ln E)^{m+1}$ for $\mathrm{Re}F_+(E,0)$, we deduce from (11) that

$$(\ln E)^m \gtrsim C^2(\ln E)^{2m+2}/(\ln E)^2 \tag{12}$$

which gives $C=0$ unless $m\leqslant 0$.

This proves the Pomeranchuk theorem in the form

$$[\sigma_+(E)/\sigma_-(E)]\to 1 \text{ as } E\to\infty, \tag{13}$$

if σ_+ or σ_- behaves like $(\ln E)^p$ with $p>0$.

It is unfortunate that this does not include the case of constant cross sections. The latter can, however, be included if an additional assumption is made about the rate of change of the amplitude in a nonforward direction. If we make the assumption that for small t in $|t|<t_0$,

$$F(E,t)\sim F(E,0)\exp\{\gamma(E)t\}, \tag{14}$$

then $\gamma(E)\leqslant C'\ln E$, if $F(E,t_0)$ is tempered. This form cannot hold for large t since it would give too fast a rate of decrease with l of partial wave amplitudes, but it agrees very well with experiment for small t. With the extra physical assumption (14), the inequality (11) gives the stronger result that $C=0$ unless $m\leqslant -1$. Using a generalized form of Leader's method,[12] one can also use assumption (14) to improve on the Froissart bound by one power of $\ln E$ to give the Froissart-Martin bound $\sigma\leqslant\ln E$. Thus unitarity with analyticity and crossing show that the Pomeranchuk theorem follows from (14) when

$$\ln E \geqslant \sigma(\text{total}) > (\ln E)^{-1+\epsilon}, \quad \epsilon>0. \tag{15}$$

It is also possible to obtain a bound on the rate of change of the difference between particle and antiparticle total cross sections. For example, if (14) holds, and

$$\sigma_\pm \sim (\ln E)^p, \quad -1<p\leqslant 1, \tag{16}$$

$$(\sigma_+-\sigma_-)\sim(\ln E)^q, \quad q<p, \tag{17}$$

then we can deduce from (6) and (11) that

$$p\geqslant 2q+1. \tag{18}$$

For example, if the total cross sections tend to constant values, their difference must decrease at least as fast as $1/(\ln E)^{1/2}$. Assuming isospin invariance, this means that the charge-exchange cross sections must decrease at the same rate.

A more general discussion of the Pomeranchuk theorem will be published elsewhere.

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STUDY OF THE REACTION $\pi^- + p \to \rho^0 + \pi^- + p$ AT 6 BeV/c†

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Two peaks in the $\rho^0\pi^\pm$ effective-mass distribution in the region between 1.0 and 1.4 BeV have recently been reported.[1] These two mass peaks, which have been named the A_1 and A_2 mesons, have been observed in π-p reactions involving four charged particles in the final state. Several theoretical interpretations[2-4] of the A_1 results have attributed this peak to the kinematic features of the π-p reaction.

The purpose of this paper is to report the results of a study of the $\rho^0\pi^-$ system in the final state of the reaction $\pi^- + p \to \pi^- + \pi^- + \pi^+ + p$ at 6 BeV/c. These results show (a) no peak in the A_1 region when background events involving $N^*(1240)$ production are removed, and (b) that the events in the $\rho^0\pi^-$ mass region between 1.0 and 1.2 BeV are consistent with peripheral production of the ρ^0 via a one-pion-exchange model. In contrast to the results for the A_1, the A_2 meson is clearly evident and is not associated with the above-mentioned type of ρ^0 production. The A_2 meson is observed at a mass of 1290 ± 10 MeV with a full width at half-maximum, Γ, of 70 ± 10 MeV. The results of a spin-parity analysis for events in the A_2 region favor $J^P = 2^-$ (p wave) or 1^+ (s wave).

Approximately 4500 four-pronged events obtained from the Brookhaven National Laboratory 80-inch liquid-hydrogen bubble chamber were scanned and measured. Using track density and χ^2 probability selection criteria, 691 of these events were fitted to the four-constraint $\pi^-\pi^-\pi^+p$ final state.

One feature of this final state at this energy is that more than 90% of the events involved a $\pi^+\pi^-$ effective mass $M(\pi^+\pi^-)$ in the ρ^0 region or a $\pi^\pm p$ effective mass $M(\pi^\pm p)$ in the $N^*(1240)$ region. The $M(\pi^+p)$, $M(\pi^-p)$, and $M(\pi^+\pi^-)$ distributions are shown in Figs. 1(a), 1(b), and 1(c). Strong N^{*++} formation is evident in the $M(\pi^+p)$ plot, with approximately 35% of the combinations falling in the region between 1110 and 1370 MeV. The $M(\pi^-p)$ plot indicates that the N^{*0} production is less strong with a much larger background present. This background is influenced, in general, by the fact that each event is plotted twice and, at higher values, by the peripheral nature of the interaction [a beamlike particle (π^-) and a targetlike particle (p) tend to have a high effective mass]. However, the production of the $N^{*0}\rho^0$ and the $N^{*0}f^0$ final states is apparent from a comparison of Figs. 1(d) and 1(e), which show the $M(\pi_2^-\pi^+)$ and $M(\pi_1^-\pi^+)$ distributions, respectively, when $M(\pi_1^-p)$ is in the $N^*(1240)$