tailed dynamics (see also Ref. 6) of the $(q\bar{q})$ system. They consider tensors of an abstract group to get symmetry breaking and only $\vec{L}\cdot\vec{S}$ to split $J=0$, 1, and 2. The detailed assignments and mass predictions differ considerably from the present one. The SU(6) singlets and $\Delta m(n',L)$ dependence of mass on dynamical quantum numbers are not considered. While the present papers were at journals, further work on $\underline{35}$ $L=1$ treated as $\tilde{U}(12)$ kinetic supermultiplets with various phenomenological mixing effects [R. Gatto, L. Maiani, and G. Preparata, Phys. Rev. 140, B1579 (1965)] has also appeared.

[12]See, for example, R. G. Newton, The Complex J Plane (W. A. Benjamin, Inc., New York, 1964).

COMPLEX INHOMOGENEOUS LORENTZ GROUP AND COMPLEX ANGULAR MOMENTUM

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The purpose of this Letter is to point out that two of the most important properties of complex angular momentum—the property of generalizing integral or half-integral angular momentum, and the property of characterizing the asymptotic behavior of the S matrix in the crossed channel—can be obtained from group-theoretical arguments.[1]

Our procedure is to construct an S matrix invariant under the complex inhomogeneous Lorentz group (ICLG).[2] That it is natural to do so follows from the fact that if the S matrix satisfies certain analyticity requirements, the Bargmann-Hall-Wightman theorem[3] guarantees that invariance under the Poincaré group implies invariance under ICLG. To the best of our knowledge, there is no reason to believe the S matrix not to be invariant under ICLG.

Assuming the S matrix to be invariant under ICLG, it is then useful to construct the irreducible representations of this group.[1] Because ICLG is not a symmetry group—only the Poincaré group $\mathcal{P}$, a subgroup of ICLG, is a symmetry group—we are free to construct nonunitary representations of ICLG, as long as a state which corresponds to a physical-particle transforms under a unitary representation of $\mathcal{P}$.[4]

A state of an irreducible representation of ICLG is characterized by a complex momentum $\Pi_\mu = P_\mu + iQ_\mu$ with complex "mass" $M^2 = \Pi_\mu \Pi^\mu$. For a given Π_μ, the state transforms under an irreducible representation of the little group for the corresponding mass. For $M \neq 0$, the little group is the homogeneous Lorentz group, whose irreducible representations are labeled by $\chi = (\rho; n)$, a pair consisting of the complex number ρ and the integer n. It is χ which plays the role of complex angular momentum.

A single, spinless "particle" is described by the state with the transformation property

$$T(\alpha_\mu, \Lambda_\nu{}^\mu)\,|\Pi^\mu\rangle_1 = \exp[i\,\mathrm{Re}(\Pi^\mu \alpha_\mu)]\,|\Lambda_\nu{}^\mu \Pi^\nu\rangle_1,$$

$$(\alpha_\mu, \Lambda_\nu{}^\mu) \in \mathrm{ICLG}. \tag{1}$$

Under the Poincaré group, it transforms unitarily by[5]

$$T(a_\mu, L_\nu{}^\mu)\,|P_\mu + iQ_\mu\rangle_1 = \exp[i(P^\mu a_\mu)]\,|L_\nu{}^\mu P^\nu + iL_\nu{}^\mu Q^\nu\rangle,$$

$$(a_\mu, L_\nu{}^\mu) \in P. \tag{2}$$

The two-particle state

$$|\Pi_\mu{}^1\rangle_1 \otimes |\Pi_\mu{}^2\rangle_1 = |\Pi_\mu{}^1, \Pi_\mu{}^2\rangle$$

can be reduced. In the center-of-mass system, it has total momentum

$$\Pi_\mu = \Pi_\mu{}^1 + \Pi_\mu{}^2 = (\Pi_0{}^1, \vec{\Pi}) + (\Pi_0{}^2, -\vec{\Pi}) = (\Pi_0{}^1 + \Pi_0{}^2, \vec{0}),$$

"mass" $S^{1/2} = [\Pi_\mu \Pi^\mu]^{1/2}$, and it contains all "spins" χ. That is, for every χ there exists a projection operator A_χ such that

$$A_\chi\,|\Pi_\mu{}^1, \Pi_\mu{}^2\rangle = |\Pi_\mu; g^{1,2}\rangle_\chi, \tag{3}$$

a state which transforms according to the irreducible representation $[S^{1/2}, \chi]$ of ICLG. $g^{1,2}$ is the complex rotation which takes a vector in the z direction into the direction $\vec{\Pi}$.

We construct the S matrix by setting[6]

$$\langle \Pi_\mu{}^3, \Pi_\mu{}^4 | S | \Pi_\mu{}^1, \Pi_\mu{}^2 \rangle = \int d\chi\, S(S,\chi) \langle \Pi_\mu, g^{3,4} | \Pi_\mu, g^{1,2} \rangle_{S,\chi}. \quad (4)$$

$\langle \Pi^\mu, g^{3,4} | \Pi^\mu, g^{1,2} \rangle_{S,\chi}$ is the analog of a scalar product in the space of the representation $(S^{1/2}, \chi)$. It is an invariant bilinear form, and depends on the scattering angle.

It can be shown that (taking equal masses for convenience) as the momentum transfer T goes to infinity (for $S \neq 0$), we obtain an explicit formula for the bilinear form in $(S^{1/2}, \chi)$, namely,

$$\langle \, | \, \rangle \sim \left| \frac{T}{S-4M^2} \right|^{\frac{1}{2}(|\mathrm{Re}\rho|-2)} \times (\text{bounded function}). \quad (5)$$

The region Ω of the complex ρ plane over which the integration in (4) is to be taken is entirely arbitrary as far as ICLG invariance is concerned. Hence, this has a more general form than the Regge representation and, moreover, some representation of this type may be expected to persist for almost any dynamics, and analytic structure of the S matrix, as long as ICLG invariance is maintained.

If Ω contains a discrete point ρ_0, integrated with discrete weight, to the right or left of the continuum contribution, then the S matrix has the Regge-like asymptotic form

$$S(S,T) \sim \left| \frac{T}{S-4M^2} \right|^{\frac{1}{2}(|\mathrm{Re}\rho_0|-2)} \times (\text{bounded function}). \quad (6)$$

By our previous assumptions, and the form of the representation with "mass" $S^{1/2}$ and "spin" χ_0, we actually know the transformation properties under $\mathcal{P}$ of that part of the S matrix responsible for this asymptotic behavior; namely, it has the transformation properties of the space $(S^{1/2}, \chi_0)$.

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[1]The assertions made in this Letter are based on results obtained in E. H. Roffman, thesis, Brandeis University, 1965 (unpublished).

[2]In Ref. 1 the results are based on the connected component of the covering group of ICLG. It is sufficient here to think of the group of transformations

$$Z_\mu \rightarrow Z_\mu + \alpha_\mu + \Lambda_\mu{}^\nu Z_\nu,$$

where Z_μ, α_μ are complex four-vectors and Λ a complex matrix with $\Lambda^T g \Lambda = g$ (here g is the usual metric tensor). We neglect symmetrization effects.

[3]D. Hall and A. S. Wightman, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 31, No. 5 (1957).

[4]This closes the gap in the argument given in several papers which attempt to explain complex angular momentum by utilizing the fact that the stationary group of the momentum-transfer vector is isomorphic to the three-dimensional Lorentz group. L. Sertorio and M. Toller, Nuovo Cimento 33, 413 (1964); M. Toller, Nuovo Cimento 37, 631 (1965); H. Joos, in Lectures in Theoretical Physics, edited by W. E. Brittin and A. O. Barut (University of Colorado Press, Boulder, Colorado, 1965), Vol. 7A, p. 132. E. H. Roffman, to be published. See also E. H. Roffman, thesis, Brandeis University, 1965 (unpublished), for a detailed discussion of the relation between the method of this Letter and the earlier work.

[5]We denote the identity representation of the little group by $|\rangle_1$. In general, the state

$$|P_\mu + iQ_\mu\rangle_\chi \text{ for } P^\mu P_\mu > 0$$

transforms under the Poincaré group $\mathcal{P}$ like the reducible representation $\sum_J \oplus |P_\mu\rangle_J$. The sum is over the representations of SU(2) contained in the representation χ of SL(2C). For certain values of χ the sum reduces to single term, and the state $|\Pi_\mu\rangle_\chi$ transforms irreducibly like $|P_\mu\rangle_J$ under $\mathcal{P}$.

[6]This is a "partial-wave expansion." $S(S,\chi)$ is the reduced S matrix.