FREDHOLM METHOD FOR BETHE-SALPETER EQUATION

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Noyes has shown[1] how an ingenious subtraction may be used to make the singly off-shell Lippmann-Schwinger equation Fredholm. Kowalski has done[2] this in the K-matrix formalism, and has extended these results to the doubly off-shell T matrix by considering the integral equations alone. This note presents a method for making the Bethe-Salpeter equation Fredholm. The method is essentially one for systematic removal of poles or logarithms in kernels of linear integral equations, when both inhomogeneous term and kernel have the same functional dependence on the parameter variables as distinct from integration variables. Define

$$D_1 = q_0^2 - \vec{q}^2 - m_1^2 + i\epsilon,$$
$$D_2 = (E-q_0)^2 - \vec{q}^2 - m_2^2 + i\epsilon,$$

and let $V(p_0\vec{p};p_0'\vec{p}')$ represent some reduced amplitude to be interated in a sum of ladder diagrams. An instance is the single-particle exchange graph,

$$V(p_0\vec{p};p_0'\vec{p}') = \frac{-g^2}{(p_0-p_0')^2-(\vec{p}-\vec{p}')^2-m^2+i\epsilon}.$$

The Bethe-Salpeter equation is

$$T(p_0\vec{p};E;p_0'\vec{p}) = V(p_0\vec{p};p_0'\vec{p}') + \int d^4q\, V(p_0\vec{p};q_0\vec{q})\frac{-i}{(D_1+i\epsilon)(D_2+i\epsilon)}T(q_0\vec{q};E;p_0'\vec{p}').$$

The equation may be expressed in partial waves[3] by putting

$$T(p_0\vec{p};p_0'\vec{p}') = \sum(2l+1)T_{p_0p,p_0'p'}P_l(\vec{p}\times\vec{p}'),$$

where the explicit energy and l dependence of T is suppressed;

$$T_{p_0p,p_0'p'} = V_{p_0p,p_0'p'} + \iint V_{p_0p,q_0q}\frac{-i4\pi q^2dqdq_0}{(D_1+i\epsilon)(D_2+i\epsilon)}T_{q_0q;p_0'p'}. \quad (1)$$

Define

$$T_{p_0p,p_0'p'} = T_{\vec{p}\vec{p}'}, \quad d\vec{q} = 4\pi q^2 dq dq_0,$$
$$\bar{p}(p_0) = +(p_0^2-m_1^2)^{1/2}, \quad p_0^2 \geq m_1^2,$$
$$= +i(m_1^2-p_0^2)^{1/2}, \quad p_0^2 < m_1^2.$$

Then $D_1(p)$ only vanishes at $p=\bar{p}$ for p positive or pure positive imaginary. Define $T_{\vec{p}\vec{p}}^{(1)}$ by

$$T_{\vec{p}\vec{p}'} = T_{p_0\bar{p},\vec{p}'} + D_1(\vec{p})T_{\vec{p}\vec{p}'}^{(1)}. \quad (2)$$

Then

$$T_{p_0\bar{p},\vec{p}'} + D_1(\vec{p})T_{\vec{p}\vec{p}'}^{(1)} = V_{\vec{p}\vec{p}'} + \iint V_{\vec{p}\vec{q}}\frac{-id\vec{q}}{(D_1+i\epsilon)(D_2+i\epsilon)}[T_{q_0\bar{q},\vec{p}'} + D_1(\vec{q})T_{\vec{q}\vec{p}}^{(1)}]. \quad (3)$$

Subtract from (3) the value of (3) at $p=\bar{p}$:

$$D_1(\vec{p})T_{\vec{p}\vec{p}}^{(1)} = V_{\vec{p}\vec{p}'} - V_{p_0\bar{p},\vec{p}'} + \iint(V_{\vec{p}\vec{q}} - V_{p_0\bar{p},\vec{q}})\frac{-id\vec{q}}{(D_1+i\epsilon)(D_2+i\epsilon)}[T_{q_0\bar{q},\vec{p}'} + D_1(q)T_{\vec{q}\vec{p}'}^{(1)}]. \quad (4)$$

Divide by $D_1(p)$ and observe that the following quantity is well defined and nonsingular:

$$\Delta(p)V_{\vec{p}\vec{p}'} \equiv \frac{V_{\vec{p}\vec{p}'} - V_{p_0\bar{p},\vec{p}'}}{D_1(\vec{p})}.$$

Hence,

$$T_{\vec{p}\vec{p}'}^{(1)}=\Delta(p)V_{\vec{p}\vec{p}'}+\iint\Delta(p)V_{\vec{p}\vec{q}}\frac{-id\vec{q}}{(D_1+i\epsilon)(D_2+i\epsilon)}[T_{q_0\bar{q},\vec{p}'}+D_1(\vec{q})T_{\vec{q}\vec{p}'}^{(1)}]. \tag{5}$$

To solve this consider the auxiliary equation

$$T_{\vec{p}\vec{p}'}^{(2)}=\Delta(p)V_{\vec{p}\vec{p}'}+\iint T_{\vec{p}\vec{q}}^{(2)}\frac{-id\vec{q}}{D_2+i\epsilon}\Delta(q)V_{\vec{q}\vec{p}'}. \tag{6}$$

Suppose, for the moment, that this has been solved to give $T_{\vec{p},\vec{q}}^{(2)}$ which is continuous on the real p_0q_0pq space (for $p,q\geqslant 0$) and on the real p_0q_0q space for $p=\bar{p}$ (real or complex) and $q\geqslant 0$. This may not in fact be the case, but it forms a starting point. Operate on (5) with

$$\iint\delta^{(2)}(\vec{p}-\vec{r})dr_0\,dr+T_{\vec{p}\vec{r}}^{(2)}\frac{d\vec{r}}{D_2+i\epsilon},$$

to get

$$T_{\vec{p}\vec{p}'}^{(1)}=T_{\vec{p}\vec{p}'}^{(2)}+\iint T_{\vec{p}\vec{q}}^{(2)}\frac{-id\vec{q}}{(D_1+i\epsilon)(D_2+i\epsilon)}T_{q_0\bar{q},\vec{p}'}. \tag{7}$$

Substitute in (3), putting $p=\bar{p}$ in (7), and use

$$R_{\vec{p}\vec{p}'}=\iint V_{\vec{p}\vec{q}}\frac{-id\vec{q}}{D_2+i\epsilon}T_{\vec{q}\vec{p}'}^{(2)}$$

to give

$$T_{p_0\bar{p},\vec{p}'}=V_{p_0\bar{p},\vec{p}'}+\iint V_{p_0\bar{p},\vec{q}}\frac{-id\vec{q}}{(D_1+i\epsilon)(D_2+i\epsilon)}T_{q_0\bar{q},\vec{p}'}+R_{p_0\bar{p},\vec{p}'}+\iint R_{p_0\bar{p},\vec{q}}\frac{-id\vec{q}}{(D_1+i\epsilon)(D_2+i\epsilon)}T_{q_0\bar{q},\vec{p}'}, \tag{8}$$

$R_{\vec{p}\vec{p}'}$ is defined both for p real and $p=\bar{p}$ when $T^{(2)}$ is defined on the real axes. The integration over the q variable may be done explicitly, leading to a factor in the kernel with a logarithmic singularity. Equation (8) is formally

$$T=V+R+\int(V+R)\kappa T dq_0.$$

This is of the type soluble by the method of Refs. 1 and 2.

Since the functional dependence of the inhomogeneous term and the kernel on p_0 [see Eq. (8)] is the same in the obvious sense, the method of substitution being described may also be used. This method may also be iterated if R is singular. This provides $T_{p_0\bar{p},\vec{p}'}$ without having to continue from real p to a possibly imaginary $\bar{p}$.

Substitution of $T_{p_0\bar{p},\vec{p}'}$ in (7) gives $T_{\vec{p}\vec{p}'}^{(1)}$ and putting both in (2) gives $T_{\vec{p}\vec{p}'}$. This solves the Bethe-Salpeter equation (3), provided, inter alia, that we know $T_{\vec{p}\vec{p}'}^{(2)}$, the solution of (6). But the method has reduced the singularity content of the original kernel by removing one pole. The method may similarly be iterated to remove other singularities systematically. In the same way, the single pole in the kernel of the Eq. (6) for $T^{(2)}$ may be removed, yielding the auxiliary equation

$$T_{\vec{p}\vec{p}'}^{(4)}=\Delta(p')\Delta(p)V_{\vec{p}\vec{p}'}+\iint\Delta(q)\Delta(p)V_{\vec{p}\vec{q}}(-id\vec{q})T_{\vec{q}\vec{p}'}^{(4)},$$

where

$$\Delta(p')\Delta(p)V_{\vec{p}\vec{p}'}=\frac{\Delta(p)V_{\vec{p}\vec{p}'}-\Delta(p)V_{\vec{p},p_0'\bar{p}'}}{D_2(\vec{p}')}.$$

This Fredholm equation has a meromorphic solution.

The most serious proviso is that $T^{(2)}$ should be analytic apart from a few singularities in the integration region. Also one requires that R be well behaved, and that $V_{p_0\bar{p},\vec{p}'}$ be defined for complex $\bar{p}$. The difficulties, and the matter of convergence, will be discussed in a forthcoming paper.

If the method is turned on the Lippmann-Schwinger equation, one obtains an interesting physical interpretation of the $(R+V)$ function. The variables are now the momenta, and

$$R_{kp'} \equiv \int V_{kq} 8\pi M q^2 dq T_{qp'}^{(2)},$$

with Lovelace's normalization[4] and M as the reduced mass. Then

$$T_{kp} = T_{pk} = \frac{R_{kp} + V_{kp}}{1 - \int (R_{kq} + V_{kq}) 8\pi M q^2 dq / (k^2 - q^2 + i\epsilon)},$$

and

$$\mathrm{Im} T_{pk}^{-1} = \frac{V_{kk} + R_{kk}}{V_{kp} + R_{kp}} (4\pi^2 M k).$$

This resembles the usual on-shell unitarity equation with a modification to k. Some other aspects of this work, including the relevance to bootstrap dynamics, will be discussed later.

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SPIN-PARITY DETERMINATION OF THE $Y_1^*(1765)$†

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Measurements of the K^-p total cross section at about 1-BeV/c incident-K^- momenta have shown a broad and asymmetric peak.[1] Further investigations led Barbaro-Galtieri, Hussain, and Tripp to suggest that two hyperon resonances with spin $\frac{5}{2}$ exist in this energy region—one an $I=0$ resonance at an energy about 1815 MeV with positive parity, the other, $I=1$ at about 1765 MeV and negative parity.[2] In this paper, we present data from the reaction $K^- + n \rightarrow \Sigma^- + \pi^+ + \pi^-$ which confirms that the $Y_1^*(1765)$ exists and that the reported spin-parity assignment, $\frac{5}{2}^-$, is correct.[3]

This study is based on 2100 of our events which fit the hypothesis $K^- + n \rightarrow \Sigma^- + \pi^+ + \pi^-$. This particular reaction has the advantage of being pure $I=1$ and having all pions visible; thus no effects from the strongly produced $Y_0^*(1815)$ are present. The data were obtained from a separated K^- beam in the Lawrence Radiation Laboratory's new 25-inch bubble chamber filled with deuterium. The incident K^- momenta were 828, 930, 1025, and 1112 MeV/c which, neglecting Fermi momentum, corresponds to a K^--n c.m. energy of 1700 to 1845 MeV.

In Fig. 1 we present the $\Sigma^-\pi^+$ invariant-mass distribution at various K^-n c.m. energies. It is evident that the reaction $K^- + n \rightarrow \Sigma^- + \pi^+ + \pi^-$

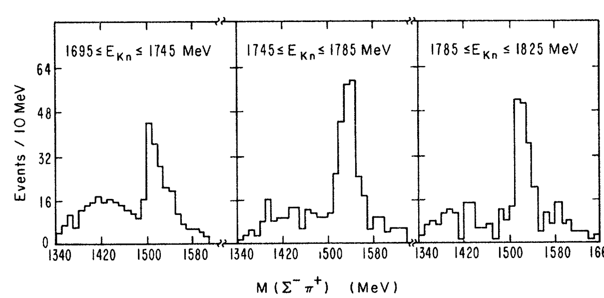

FIG. 1. Invariant mass of the $\Sigma^-\pi^+$ system produced in the reaction $K^- + n \rightarrow \Sigma^- + \pi^+ + \pi^-$.