LOCAL REPRESENTATIONS AND MASS SPECTRUM

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In a series of papers, O'Raifeartaigh[1] claimed an interesting negative result, which states given any real Lie algebra containing the Poincaré Lie algebra $\mathcal{P}$, and an irreducible representation of it in an Hilbert space $\mathcal{K}$ so that $P_\mu P^\mu$ is self-adjoint, then if there exists one eigenvalue m of $P_\mu P^\mu$, it is all the spectrum. This result was criticized by the authors,[2] who showed an error in the proof. Recently, Jost[3] was able to prove the following much weaker result: Given a unitary continuous irreducible representation of a connected Lie group containing the Poincaré group on $\mathcal{K}$, the spectrum of $P_\mu P^\mu$ is a connected set (thus, if there is an isolated eigenvalue, it is all the spectrum). All hypotheses are needed in the proof, though not dictated by physical necessities. It is the aim of this Letter to present a counter example which might be of physical interest to the original conjecture, and thereby to clarify the situation.

Let us first explain some possible motivations: (a) In the present state of the unification problem, there is no reason to prefer a group structure to a Lie-algebra one for internal or unified symmetries. (b) In this counter example, we shall deal with local representations (i.e., of Lie algebras). Its physical sense can be understood by a very simple analogy: For a free electron in one dimension, the Hamiltonian $p^2/2m$ is represented by a self-adjoint operator in $L^2(-\infty,\infty)$ which has a continuous spectrum; if there is an infinite potential well between a and b, we know we have a discrete spectrum that can equivalently be obtained by taking the spectrum of the free Hamiltonian self-adjoint in $L^2(a,b)$ with the suitable boundary conditions.

And now to the example itself. Let su(2,2) be the Lie algebra of the conformal group. Consider the formally skew-adjoint operators

$$M_{\mu\nu}=x^\nu\partial_\mu-x^\mu\partial_\nu,$$

$$P_\mu=\partial_\mu$$

$$A_\mu=x^2\partial_\mu-2x^\mu x_\rho\partial_\rho-4x^\mu$$

$$=\tfrac{1}{2}(x^2\partial_\mu+\partial_\mu x^2)-(x^\mu x_\rho\partial_\rho+\partial_\rho x_\rho x^\mu),$$

$$A_0=x_\rho\partial_\rho+2=\tfrac{1}{2}(x_\rho\partial_\rho+\partial_\rho x_\rho), \tag{1}$$

where $x^2=x_\mu x^\mu$, $x^\mu=g^{\mu\nu}x_\nu$, $\partial_\mu=\partial/\partial x_\mu$ ($\mu,\nu=1,2,3,4$). When these operators act (e.g.) on distributions, we check easily that they verify the commutation relations of su(2,2) (cf., e.g., Flato, Sternheimer, and Vigier, and Bohm, Flato, Sternheimer, and Vigier[4]). Moreover, this defines an algebraically irreducible representation of su(2,2) on the vector space $\mathcal{C}[x_1, x_2, x_3, x_4]$ of polynomials in four unknowns (the A_μ raise the degree, the P_μ lower it, and the $M_{\mu\nu}$ and A_0 leave it unchanged)—we get semireducible representations of $\mathcal{P}$ when we represent it by $(M_{\mu\nu},A_\mu)$ or $(M_{\mu\nu},P_\mu)$.

Now let $\mathcal{K}$ be a topological vector space of (generalized or not) functions, in which the polynomials (properly weighted if necessary) form a dense subspace [contained in the domains of all operators of (1) if they are not defined everywhere]. By similar arguments (at least if the possible weight is a polynomial), we see that (1) defines a topologically irreducible representation (i.e., there is no proper closed subspace of $\mathcal{K}$ on which the operators are densely defined and which is invariant). In the following, we shall consider the space $\mathcal{K}=L^2(Q)$, where Q is a bounded domain in R^4, more precisely in our case the cube $Q:\{0\le x_\mu\le a\}$ of boundary $\partial Q:\{x_\mu=0 \text{ or } a,\ 0\le x_{(\mu)\nu}'\le a\}$, where we define the four three-component vectors $\vec{x}_{(\mu)}'=(x_{(\mu)\nu}')$ ($\mu\ne\nu$, $x_{(\mu)\nu}'=x_\nu$), so that the four-component vector $x=(x_\mu)$ can be written $x=(x_1,\vec{x}_{(1)}')$, for instance.

One checks easily that (1) defines skew-symmetric operators on the domain of absolutely continuous (ac) functions, with L^2 derivatives, vanishing on ∂Q; more generally, all finite-order operators of the enveloping algebra of $i$su(2,2) (multiply all generators by i) are symmetric (Hermitian) on the (dense) domain $\mathcal{C}_0^\infty(Q)$ of infinitely differentiable functions with compact support inside Q, when considered as differential operators,[5] and have self-adjoint extensions that are restrictions of the same differential operators acting in the distribution way.

We shall now restrict our attention to the mass operator. It is clear[6] that $-\partial_\mu{}^2$ is self-adjoint and positive on the domain $D(P_\mu^2)$ of functions f in $L^2(Q)$ with $\partial_\mu f$ ac with respect to x_μ, $\partial_\mu^2 f \in L^2(Q)$, and with the boundary conditions $f(0, \vec{x}_{(\mu)}') = f(a, \vec{x}_{(\mu)}')$, $\partial_\mu f(0, \vec{x}_{(\mu)}') = \partial_\mu f(a, \vec{x}_{(\mu)}')$; as such, it is the square of the operator $i\partial_\mu$ on $D(P_\mu) = \{f \in L^2;\ f \text{ ac in } x_\mu,\ \partial_\mu f \in L^2,\ f(0, \vec{x}_{(\mu)}') = f(a, \vec{x}_{(\mu)}')\}$. [If we take $f(0, \vec{x}_{(\mu)}') = f(a, \vec{x}_{(\mu)}') = 0$ for $D(P_\mu^2)$, we get an operator which is not the Hilbert-space square of $i\partial_\mu$ on $D(P_\mu)$, but is still $-(\partial/\partial x_\mu)^2$; in this case, we shall have functions vanishing on ∂Q for the domain of $\Box$,

$$\prod_{\mu=1}^{4} \sin\left(\frac{\pi}{a} n_\mu x_\mu\right)$$

with $n_\mu \geq 1$ (without summation) for eigenfunctions, and $(\pi/a)\sqrt{N}$ instead of $(2\pi/a)\sqrt{N}$ for the levels.]

We can thus define the symmetric operator $\Box = \partial_4^2 - (\partial_1^2 + \partial_2^2 + \partial_3^2)$ on the common domain

$$D_0 = \bigcap_{\mu=1}^{4} D(P_\mu^2).$$

Now, any function $f \in L^2(Q)$ has a Fourier development

$$f \sim \sum A_{\vec{n}} \exp\left(\frac{2i\pi}{a} n_\mu x_\mu\right),$$

where $\vec{n} = (n_\mu)$, $n_\mu \in Z$ (integer); if $f \in D_0$, we have equality, and moreover

$$\Box f \sim \left(\frac{2\pi}{a}\right)^2 \sum n_\mu n^\mu A_{\vec{n}} \exp\left(\frac{2i\pi}{a} n_\mu x_\mu\right).$$

Let $E(m)$ denote the projector on the subspace $\mathcal{H}_m$ generated by the $\exp[(2i\pi/a)n_\mu x_\mu]$ for $n_\mu n^\mu = m \in Z$ (any integer can be written, in many ways, as $n_\mu n^\mu$); $E(m)\mathcal{H} = \mathcal{H}_m$, and $E(m)$ is a resolution of identity (cf. Ref. 5). Therefore

$$M = (2\pi/a)^2 \sum_{m=-\infty}^{\infty} mE(m)$$

is a well-defined self-adjoint operator with domain

$$D(M) = \left\{f \sim \sum A_{\vec{n}} \exp\left(\frac{2i\pi}{a} n_\mu x_\mu\right); \sum |n_\mu n^\mu A_{\vec{n}}|^2 < \infty\right\} \supset D_0.$$

M coincidences with $\Box$ on D_0; moreover, it is clearly the closure of $\Box$, which is therefore essentially self-adjoint on D_0 (and has the same spectrum).

$P_\mu P^\mu$ is therefore here represented by a self-adjoint operator M with a purely discrete spectrum, consisting of isolated eigenvalues. The formal calculation of O'Raifeartaigh[1] is here pointless as it is justified only on the space $\mathcal{H}_m{}^e$ (cf. Ref. 2) which is $\{0\}$—in fact, the $M_{\mu\nu}$, A_μ, and A_0 do not leave $\mathcal{H}_m$ invariant, whenever they are defined on it. Moreover, Jost's hypotheses[3] do not apply, since we cannot get from (1) a unitary representation of the group SU(2, 2), because of lack of analytic vectors (though one-parameter groups can be defined by means of spectral resolution).

In addition, we get the "mass formula" $m = N^{1/2} m_0$, where N is an integer [and $m_0 = (2\pi/a)$]. If we take, for instance (as in Sternheimer,[7] where $N^{1/2}$ is an integer), the pion mass $m_\pi = [\frac{1}{2}(m_{\pi^+}{}^2 + m_{\pi^0}{}^2)]^{1/2}$ as m_0, we get an experimentally well-verified formula (there are no counter examples, although there are many gaps, and the formula is not very significant for large N).

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