Table I. Results of analytic continuation of the off-shell T matrix for the Yukawa potential of strength $g = -2$ at $k^2 = 0$.

Order of fitting N	$\alpha^{-1} = \tan\delta/k$	β/k
Born approximation	+2.0	0.0
1	−8.0605	0.9577
2	−8.3048	0.9351
3	−7.9701	0.9865
4	−7.9432	0.9928
5	−7.9870	0.9833
Exact	−7.9114	1.0000

should equal +1 (unitarity!). In Table I are shown results of this extrapolation for the potential of strength $g = -2$ at $k^2 = 0$. For this value of g there is a bound-state pole in $T(w)$ at very small negative w; and our extrapolation around this pole represents a very tough test of the present method. We achieved similarly good results at several values of g and k^2, the agreement between calculated and known results being 1% or better. (The unitarity condition $\beta/k = 1$ can serve as a measure of the accuracy in cases where the exact answer is not known.)

Still lacking is some firm mathematical understanding of the convergence of this analytic continuation, and the dependence on the location and accuracy of the input numbers. Nevertheless, the success we have found with our first crude attempts convinces us that the method is basically sound; and we look forward to a broad range of applications as well as a refinement of the numerical techniques.

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[1]See T.-Y. Wu and T. Ohmura, Quantum Theory of Scattering (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1962), Sec. D.

[2]This construction is reminiscent of the Padé method; see G. A. Baker's review article in Advances in Theoretical Physics (Academic Press, Inc., New York, 1965), Vol. 1. However, that approach relies entirely on the Born series, while we have no such bias.

CURRENT-COMMUTATOR CONSTRAINTS ON THREE- AND FOUR-POINT FUNCTIONS*

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The equal-time commutator algebra of quark currents,[1] combined with a causal representation for vertex functions or forward scattering amplitudes such as the Deser-Gilbert-Sudarshan[2] (DGS) formula, provides constraints on these amplitudes.[3] There are numerous applications of these constraints, some of them studied independently by Bjorken[4] especially in connection with electromagnetic renormalization. In this note, we discuss the general theory briefly, and give the constraints in the form of sum rules in the vertex function VVP (V = vector particle, P = pseudoscalar).

The DGS representation for causal commutators reads

$$(2E)^{1/2}\langle p|[J(x),J'(y)]|0\rangle = \int d\lambda^2 d\beta d^4q \frac{\epsilon(q_0)}{(2\pi)^3} e^{-iq(x-y)-ipy+i\beta p(x-y)} \times H(\lambda^2,\beta)\delta(q^2-\lambda^2), \tag{1}$$

where p refers to a single-particle state of momentum p (energy E). The same spectral function H appears in the matrix element of the time-ordered product:

$$(2E)^{1/2}\langle p|T(J(x)J'(y))|0\rangle = \int \frac{d\lambda^2 d\beta d^4q}{i(2\pi)^4} \frac{e^{-iq(x-y)-ipy+i\beta p(x-y)}}{q^2-\lambda^2+i\epsilon} \times H(\lambda^2,\beta). \tag{2}$$

These are support conditions on H: $0 \geqslant \beta \geqslant -1$, $\lambda^2 > 0$ (in the absence of zero-mass intermediate states). H may also depend on p^2, but this dependence will not be made explicit. It is understood that the integral over d^4q is to be done first.

Now consider the VVP three-point function for a pion of momentum p. We write for the invariant amplitude

$$M(k,p) = \epsilon^{\mu}(k)\epsilon^{\nu}(p-k)M_{\mu\nu}(k,p),$$

$$M_{\mu\nu}(k,p) = \epsilon_{\mu\nu\alpha\beta}p^{\alpha}k^{\beta}M(k,p). \tag{3}$$

The ϵ^{μ} are polarization vectors. It is sufficient to study only the spacelike components (denoted by Latin indices), since current conservation and Lorentz invariance completely determines the whole matrix element from its spacelike components. We are interested in three cases: (1) $\pi^0 \to 2\gamma$; (2) $\omega \to \pi^0+\gamma$; (3) $\omega \to \rho+\pi$. All three cases will be exemplified by π^0 decay. Notice that the kinematic structure of the VVP vertex is such that the decay amplitude coincides with the time-ordered product, with no Schwinger terms[5] (gradients of delta-functions) needed. This follows immediately from the DGS representation, which is very useful for studying Schwinger terms. For π^0 decay, the usual reduction formula gives

$$M_{ij} = -ie^2(2E)^{1/2}\int d^4z\, e^{ikz}\langle p|T(J_i(z), J_j(0))|0\rangle, \quad (4)$$

where J is the electromagnetic current, coupled with charge e. The function H in the DGS representation (2) is replaced by a function H_{ij}, which can only have the form

$$H_{ij} = \epsilon_{ij\alpha\beta}\, p^{\alpha} q^{\beta} H(\lambda^2, \beta), \quad (5)$$

$$M(k,p) = e^2\int d\lambda^2 d\beta \frac{H(\lambda^2,\beta)}{(\beta p+k)^2-\lambda^2+i\epsilon}, \quad (6)$$

and the mass-shell decay amplitude is

$$M = e^2\int \frac{d\lambda^2 d\beta\, H(\lambda^2,\beta)}{\beta(\beta+1)m_{\pi}^2-\lambda^2+i\epsilon}. \quad (7)$$

One finds in this case the support condition $\lambda^2 > 4m_{\pi}^2$, so the term $\beta(\beta+1)m_{\pi}^2$ can be dropped without appreciable error. Now let us take seriously the commutation rules for the spacelike components of quark current densities,[1] so that

$$\langle p|[J_i(x), J_j(y)]|0\rangle_{x_0=y_0} = \frac{2i}{3}\epsilon_{ijk}\langle p|J_k^{5(3)}(x)|0\rangle\delta_3(\vec{x}-\vec{y}). \quad (8)$$

Terms involving the vector current, and other terms in the axial-vector current, do not contribute to this matrix element. The reader need hardly be warned that these commutation rules for other than time components of currents may well be false, even though the commutation rules for the charges (which lead to the Weisberger-Adler[6] sum rules) may be valid. It is important to notice, however, that a gradient of a delta-function may not appear in the matrix element (8), because of the kinematical structure. In (8), $J_k^{5(3)}$ is the third component (in isospace) of the axial-vector current. It is known that

$$(2E)^{1/2}\langle p(a)|J_k^{5(b)}(0)|0\rangle = -ip_k a_{\pi}\delta_{ab}, \quad (9)$$

where a_{π} is determined from the leptonic decay mode of the charged pion. Using (5) in the DGS formula (1) at equal times, and comparing to (8) and (9), we find

$$\int d\lambda^2 d\beta\, H(\lambda^2,\beta) = \tfrac{2}{3}a_{\pi}, \quad (10)$$

since the time component of q in H_{ij} gives a derivative at equal times of the free-field commutator function, and hence a delta function. We now have a constraint on the weight function appearing in the representation (6) for the vertex function. Apparently the only direct test of this constraint is for $k^2 \gg k\cdot p, p^2$, in which case

$$M(k,p) \sim 2a\pi/3k^2, \quad k^2\to\infty. \quad (11)$$

One would have to study the emission of soft pions from internal photons of very large mass, as might occur in electron-electron or electron-proton scattering experiments at large momentum transfers.

As an additional test of (10), we study two models of π^0 decay: the old nucleon-loop ($\pi \to \bar{N}+N \to 2\gamma$) model,[7] and the vector-meson dominance model of Gell-Mann, Sharp, and Wagner[8] (GSW). It turns out that in neither of these models is the constraint satisfied, because the asymptotic behavior at large k^2 is wrong. In the nucleon-loop model, the asymptotic behavior is $(\log k^2)/k^2$, contradicting (11). The GSW model, in its simplest version, has an amplitude which is a product of two vector meson poles:

$$M(k,p) = \frac{C}{(k^2-m_{\rho}^2)[(k-p)^2-m_{\omega}^2]} + (\rho \leftrightarrow \omega). \quad (12)$$

The weight function H can be constructed explicitly, and one finds that the integral in (10) is zero, as is obvious because the behavior at large k^2 is like $1/k^4$, again contradicting (11). This illustrates a general feature which renders the usefulness of the constraint quite dubious in many cases: The mass-shell amplitude (k^2 not large) may be dominated by a term that decreases faster than $1/k^2$ at large k^2, and thus does not contribute to (10). For the sake of argument, let it be assumed that this is not the case. For instance, if one merely inserts ρ and ω intermediate states into the commutation relation (8), one gets a sum of poles,[9] rather than a product, and the amplitude has the required asymptotic behavior. Also in certain amplitudes where it is reasonable to assume that two-particle intermediate states dominate the dispersion relations, H is essentially of one sign, and the integral (10) cannot vanish. Then one can replace the denominator in (7) by some effective value of λ^2, presumably of the order of $m_\rho^{\ 2}$ or $m_\omega^{\ 2}$, and write

$$M \sim e^2 \int d\lambda^2 d\beta\, H(\lambda^2,\beta)\langle\lambda^{-2}\rangle = \tfrac{2}{3}e^2 a_\pi \langle\lambda^{-2}\rangle. \tag{13}$$

Taking $\langle\,\rangle = m_\rho^{\ -2}$, the predicted π^0 width is about 1.5 eV, compared to the experimental width of about 3 eV.[10] The decay $\omega \to \pi^0+\gamma$ can be handled in the same way, taking into account ω-φ mixing, which as in the π^0 case is assumed to have the consequence that $\varphi\to 3\pi$ is forbidden. The invariant matrix element is

$$M = e\gamma a_\pi \langle\lambda^{-2}\rangle, \quad \gamma^2/4\pi = 2, \tag{14}$$

and the branching ratio is

$$\Gamma(\pi_0 \to 2\gamma)/\Gamma(\omega\to\pi+\gamma) = 1.7\times 10^{-5}R^2, \tag{15}$$

where R is the ratio of $\langle\lambda^{-2}\rangle$ for the two decays. If R is taken to be unity, then there is no difference between the present model for the ratio (15) and the Dashen-Sharp[11] model or SU(6)[12] model. Experimentally,[10] the branching ratio is about $(0.6\text{-}0.3)\times 10^{-5}$, a discrepancy of a factor of from 3 to 6. In the decay $\omega\to\rho+\pi$, if one uses the same value for $\langle\lambda^{-2}\rangle$, the branching ratio $\Gamma(\omega\to\pi+\gamma)/\Gamma(\omega\to\rho+\pi)$ is 0.17, just as in the GSW model. The experimental ratio is 0.14, in excellent agreement. There seems to be no particular reason why $\langle\lambda^{-2}\rangle$ should be the same for $\pi_0\to 2\gamma$ and $\omega\to\pi+\gamma$. It is known that the effective pole position in isovector-nucleon form factors is at about $t=15$, rather than at $t=m_\rho^{\ 2}=29$.[13] If one simply replaces $\langle\lambda^{-2}\rangle$ for isovector channels by 1/15, but $\langle\lambda^{-2}\rangle = 1/30$ for isoscalar channels, then the π^0 width is raised to 3.4 eV, and (15) is changed to 0.9×10^{-5}. In any event, the point is that reasonable values for $\langle\lambda^{-2}\rangle$, and use of the constraint (10), lead to reasonable values for the π^0 lifetime.

Other applications of the above formulas to processes like $\eta\to 2\gamma$ and $\rho\to\pi+\gamma$ immediately come to mind. However, precise experimental information is lacking as yet for these processes.

As an addendum, we point out the utility of causal representations, such as the Lehmann representation for propagators and the DGS formula, for studying the appearance of gradients of delta-functions in commutators. The vacuum expectation value of the commutator of two conserved currents is very easily handled,[4,14] and gives a gradient term appearing in the commutator of a time component and a space component, proportional to the integral of the spectral function. If the state p in (1) or (4) refers to a scalar (rather than pseudoscalar) state, it is easy to demonstrate that there must be a term in $H_{\mu\nu}$ proportional to $q_\mu q_\nu$, which gives rise to gradient terms in the commutator of a space and a time component multiplied by the integral of the spectral function. But it may be that the gradient terms are c numbers rather than operators, so that their nondiagonal components must vanish. It would then be required that the integral of the spectral function vanish. Some other applications of these techniques to forward scattering amplitudes will be reported in future work.

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OBSERVATION AND PROPERTIES OF THE NEUTRAL A_2 MESON*

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From a 250 000-picture exposure of 3.65-BeV/c π^+ mesons in the Brookhaven 20-inch bubble chamber, we have obtained 2993 events of the type

$$\pi^+ + d \rightarrow \pi^+ + \pi^- + \pi^0 + p + p$$

with both protons visible. The three-pion mass spectrum shows an enhancement at about 1300 MeV which we interpret as the $A_2^0(1300)$ meson. Further analysis of the data shows its subsequent decay into $\rho^\pm\pi^\mp$. The properties of the A_2^0 appear to be consistent with those expected for the neutral member of the A_2 triplet, the charged members of which have been observed[1,2] in the reactions $\pi^\pm + p \rightarrow A_2^\pm + p$. The branching ratios of the A_2^0 indicate an $I = 1$ state which strongly prefers $\rho\pi$ (as opposed to direct 3π) decay. The Dalitz plot density favors a spin-parity assignment of 2^+.

There is no substantial peaking in the $A_1(1080)$ mass region. A comparison with $\pi^- p$ data indicates that the A_1 and A_2 may have different production mechanisms. This can be interpreted as support for the idea that a kinematical (e.g., Deck) effect is present in A_1 production.[3]

Event selection. – The events were analyzed using the TRED and GRIND programs and have χ^2 probability and track ionizations consistent with the hypothesis $\pi^+ + d \rightarrow \pi^+ + \pi^- + \pi^0 + p + p$. The spectator protons (defined in each event as the proton of lower momentum) have a range of 1 mm or greater. Of these spectators, 28% have a momentum greater than 250 MeV/c which exceeds our prediction of 9%, based on the Hulthén wave function for the deuteron. No significant correlation was found between the momentum of the spectator proton and the characteristics of the three pion states.

An additional 550 events having a possible proton track with momentum greater than 1.7 BeV/c but also fitting $\pi^+ + d \rightarrow \pi^+ + \pi^+ + \pi^- + p + n$ have not been included. This possible bias against events with high Δ^2 (four-momentum transfer squared) from the deuteron to the pp system does not materially affect the reactions discussed here, which are primarily low-Δ^2 processes.

General mass spectra. – The invariant mass of the $\pi^+\pi^-\pi^0$ system is shown in Fig. 1. In addition to the η^0- and ω^0-meson peaks, a broad enhancement is seen around 1300 MeV. Previous studies[4] of this peak in similar experiments have not been able to demonstrate that it was substantially due to $A_2^0 \rightarrow \rho^\pm + \pi^\mp$. The lower histogram and smooth curve are discussed later in the text.

The two-body pion-pion and pion-proton mass plots (not shown) give evidence for ρ^+, ρ^0, and ρ^- production and for production of the two-body final states $\rho^+ + N^{*0}(1238)$ and $\rho^0 + N^{*+}(1238)$. There is no evidence for the $\rho^- N^{*++}$ final state. This is perhaps explained by the fact that ρ^+ and ρ^0 can be produced in $\pi^+ n$ collisions by one-pion exchange but ρ^- cannot. The Δ^2 distribution shapes and total numbers of events for the

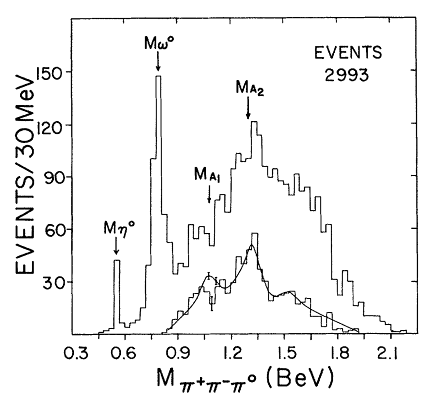

FIG. 1. Effective mass of $\pi^+\pi^-\pi^0$ for all events from $\pi^+ + d \rightarrow \pi^+ + \pi^- + \pi^0 + p + p$. The subsample of our events with ρ^+ or ρ^- and $\Delta^2(\rightarrow 3\pi) \leq 0.85$ (BeV/c)^2 is shown with a smooth curve taken from a compilation of $\pi^\pm + p \rightarrow \rho^0 + \pi^\pm + p$ data.